THE PHYSICS OF TIME TRAVEL

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Abstract

Using the Principle of General Covariance, the metric, the affine connection and the relativistic equations of motion of a fluid in an accelerated reference frame are obtained from the coordinate transformations between inertial and accelerated observers. Firstly, it is assumed that the Gravitational field created by the fluid is negligible, which implies absence of gravity and flat space-time. The great advantage is that the coordinate transformations are explicit in the relativistic equations of motion as variables and so they can be obtained by solving them. The available degrees of freedom fix the independent variables and the rest will be functions of them. In the case of rotational motion this means that the time coordinate transformation depends on the only degree of freedom, that is, the angular velocity of the fluid. In this way, the angular velocity can be chosen to achieve that time coordinate transformation is a function that allows the perfect fluid the travel to the past, at a rate consistent with time dilation, checking that the external force applied is finite and discussing if this solution is physically possible. In the Appendix the formalism is generalized to the case in which gravity is present.

Keywords: classical general relativity, flat space-time, accelerated frames, coordinate transformations, time travel, scalar field, dark energy.

1. Introduction

There are basically three ways in which time travel arises in General Relativity: rotating space-times containing Closed Timelike Curves (CTCs) [1-5], other geometries with CTCs [6-11] and space-times which allow for faster than light travel to generate CTCs [9-10], [12-14]. These models present several practical difficulties depending on the particular CTC containing geometry: the Tomimatsu-Sato geometries in vacuum involve a naked singularity, the CTC region in the Kerr space-time is confined behind an event horizon, superluminal travel often require substantial quantities of exotic matter [15], [16] and other models need infinitely large distributions of matter.

However, there is an alternative based on the Principle of Equivalence of Gravitation and Inertia postulated by Einstein in the General Relativity. As we know, it is easier to handle inertia appearing with the accelerated motion than gravity, although the effects in both cases must be locally equivalent. Due to this, it is possible to find metrics originated as a consequence of accelerated motion to travel to the past.

Unfortunately, there are different ways to deal with accelerated motion in relativity [17-27] and there is no general agreement on how to write the metric tensor of the accelerated reference frame not even the coordinate transformations between the inertial and accelerated frames. There is no agreement either on how to establish the relativistic transformation to rotating frames. The paper of Klauber reviews different theories of relativistic rotation [28].

Although there is no agreement either on the formulation of the Principle of General Covariance [29], we follow the definition given in [30] to obtain the relativistic equations of motion of the fluid in the accelerated reference frame from the coordinate transformations between inertial and accelerated frames. Doing this, the coordinate transformations appear as variables in the relativistic equations of motion and so they can be obtained by solving them. In order to solve the relativistic equations of motion we need to fix the gauge, that is, the reference frame.

In the case of rotational motion, we will use the same spatial coordinate transformations between the inertial and rotating frames as in [28] and [31-36] to obtain the metric, the affine connection and the relativistic equations of motion in the latter. The time coordinate transformation will be determined by the gauge. The gauge will be chosen so that the fluid remains at rest with respect to the rotating reference frame and to achieve time dilation, which guarantees validity to relativistic rotational velocities. Finally, we obtain the density of the external force acting on the fluid from the relativistic equations of motion. It is important to note, that all this is equivalent to finding solutions of the relativistic equations of motion that depend on the only degree of freedom, that is, the angular velocity of the fluid as we will see in section 4.1.

This paper shows the possibility of travelling to the past using a simple model of rotating perfect fluid without pressure. Firstly, it is assumed that the Gravitational field created by the fluid is negligible, which implies flat space-time. The time coordinate transformation between the inertial and rotating frame will be depend on the only available degree of freedom, that is, the angular velocity of the fluid, so this can be chosen so that the fluid goes back in time. Finally, the external force applied on the fluid is calculated from the relativistic equations of motion checking that it is always finite and discussing if this solution is physically possible.

The main advantage is that in this flat space-time CTCs are not needed to go back in time. Instead of that, the angular velocity of the fluid is chosen so that the time coordinate transformation between the inertial and rotating frames is a pointy function. The function has this shape to allow the travel to any time in the past and to be compatible with time dilation. This is so, because the travel to the past in a flat space-time must be understood as follows: on the rotating and inertial frames, time always goes forward. However, if the clocks on the rotating frame S could be compared with clocks on the inertial frame S´, the latter would go back at a rate determined by time dilation. In order to achieve this time reversal, the function must go from increasing to decreasing so that it must reach a maximum, which must be pointy to be compatible with time dilation. Other advantages are: neither huge amounts of matter that curve space-time nor exotic matter or negative energy are needed and also the mass-energy is conserved in the Lorentzian frame of the observer at rest.

However, the shape of this function implies that the application is not a diffeomorphism which could cause physical problems. Nevertheless, at the time in which the derivative diverges, due to the redshift, the wavelength of a photon emitted in S is infinite for the observer S´, and its energy is zero, so it can't be detected.[1] The same is true for any radiation. In addition, during the travel to the past, since the time on the inertial reference frame S´ is reversed, any signal emitted from S to S´ is propagated in the opposite direction for an observer in S´ and it never reaches him. The same happens for signals emitted from S´ to S from the point of view of the rotating observer. This is something similar to what happens inside a black hole but in our case, the metric has a time singularity as we will show in Section 5.2. As a result, both observers cannot measure the same events and they can't compare their clocks.[2] For these reasons, none of them is aware that the time reversal takes place and there are no contradictions.

Finally we consider the case when the Gravitational field created by the fluid is important and therefore gravity is present.

Sections 2 to 5 consider the motion in a flat space-time without gravity. Section 2 studies moving reference frames and the inertial field, explaining the differences between gravity and inertia and the meaning of time. Section 3, analyzes rotating reference frames. In section 4, the previous results are applied to obtain the relativistic motion of the rotating perfect fluid without pressure. In section 5, the force applied on this model of rotating fluid is calculated to go back in time discussing if this solution is physically possible. Finally, the Appendix connects with the equations of the field and gravity.

2. Moving reference frames

There is no agreement on how to write the coordinate transformations between the inertial and accelerated frames. For this reason, the coordinate transformations used here to construct the metric are presented in a general form. Moreover, we will use a different gauge to constrain the available degrees of freedom and to achieve time dilation.

2.1. Equations of motion

[1] This is valid, at that time, even if there is no time reversal and the fluid continues travelling to the future as indicated in the previous paragraph.
[2] In general, due to the difficulty to synchronize accelerated clocks with inertial clocks, the clocks of both frames must be synchronized before starting the motion, never after, as indicated in section 5.1.

When relativistic dynamics is studied, it is convenient to define two observers linked to two different reference frames: one that is considered at rest and the other moving with respect to the first. From now on they are going to be known as reference frame at rest S´ and moving reference frame S.

Let S´ be an inertial reference frame in which the fluid is initially at rest and the laws of Special Relativity are valid globally or at least within a finite volume. In this frame the metric is the Minkowski metric[3]

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta_{\nu\mu}. \tag{1}$$

An observer located in S´ defines Cartesian coordinates $x'^{\mu} = (t', x', y', z')$ to locate events. Let S be a moving reference frame with respect to S´ in which other coordinates $x^{\mu} = (x^0, x^1, x^2, x^3)$ are used to locate events.

The equations of motion in S´ of a fluid moving under the action of an external force are according to the laws of Special Relativity [30]

$$f'^{\mu} = \frac{\partial T'^{\mu\alpha}}{\partial x'^{\alpha}}. \tag{2}$$

The equations of motion must be invariant under any coordinate transformations according to the Principle of General Covariance set out in [30]. The coordinate transformations between S´ and S are functions of the form $x'^{\mu} = x'^{\mu}(x^0, x^1, x^2, x^3)$. The only condition imposed is that they must be invertible. The inverse transformations are functions of the form $x^{\mu} = x^{\mu}(t', x', y', z')$. The necessary and sufficient condition for this to be true is that the Jacobian of the transformation is nonzero.

Let's suppose that the coordinate transformations are global continuous functions, over all space-time or at least the volume of the fluid. Then, carrying out any coordinate transformations from S´ to S, in the equations of motion (2) , we can find the equations of motion of a fluid in any reference frame, for example S

$$f^{\mu} = T^{\mu\alpha}{}_{;\alpha} = \frac{\partial T^{\mu\alpha}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\alpha\beta} T^{\mu\beta} + \Gamma^{\mu}_{\alpha\beta} T^{\alpha\beta} \tag{3}$$

where

$$f^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} f'^{\alpha}. \tag{4}$$

is the density of the external force and $T^{\mu\nu}$ is the energy-momentum tensor. The metric is

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \tag{5}$$

and the affine connection is

[3] From now on, a system of units in which the speed of light is unity ($c = 1$) is used and any index that appears twice, once as a subscript and once as a superscript is understood to be summed over.

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \tag{6}$$

or

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} + \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right). \tag{7}$$

Moreover

$$\frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu}. \tag{8}$$

The principle of General Covariance set out in [30] says that equations of motion (3) are true in any reference frame in the absence of gravity, that is in a flat space-time as we will show in section 2.3. In practice, this happens when the gravitational field created by the fluid is so weak that it barely curves space-time. In addition, they are the same as in a true Gravitational field in the presence of external forces [30], [37] according to the Principle of Equivalence. It is important to note that the equations of motion (3) explicitly show the coordinate transformations between S´ and S through the affine connection Eqs. (6) and so the coordinate transformations appear as variables, so they can be obtained by solving them.

In order to solve the relativistic equations of motion (3) we need to fix the gauge, that is, the reference frame. If the reference frame S is chosen so that the fluid remains at rest with respect to it, then

$$\frac{dx^i}{d\tau} = 0. \tag{9}$$

Equations (9) do not fix the gauge alone. It is necessary to add another condition: the time-time component of the metric has to be invariant

$$g_{tt} = \eta_{tt} = -1. \tag{10}$$

This condition is needed for both the fluid and the reference frame S comove in time.

2.2. Meaning of time

As it is well known, the three spatial dimensions are characterized by freedom of motion. This is not so in the case of time as it is demonstrated below.

Equation (10) and the time-time component of Eqs. (5) give

$$\left(\frac{\partial x'}{\partial t}\right)^2 + \left(\frac{\partial y'}{\partial t}\right)^2 + \left(\frac{\partial z'}{\partial t}\right)^2 = -1 + \left(\frac{\partial t'}{\partial t}\right)^2. \tag{11}$$

On the other hand, the components of the velocity of S with respect to S´ are

$$v'^i = \frac{dx'^i}{dt'} = \frac{\frac{\partial x'^i}{\partial t}}{\frac{\partial t'}{\partial t}} \tag{12}$$

since S is at rest in its own reference frame.

In addition, the magnitude of the velocity is calculated with Eqs. (12) and (11)

$$v' = \sqrt{v'^i v'^i} = \sqrt{\frac{-1+\left(\frac{\partial t'}{\partial t}\right)^2}{\left(\frac{\partial t'}{\partial t}\right)^2}} \leq 1. \tag{13}$$

This equation shows, as expected, that it can never be greater than unity, that is, any reference frame can't move at a speed greater than the speed of light, with respect to S´. From Eq. (13), it is found

$$\frac{\partial t'}{\partial t} = \frac{1}{\sqrt{1-v'^2}} = \gamma'. \tag{14}$$

This equation explains time dilation when velocity increases which guarantees validity to relativistic velocities and for this reason, time dilation serves as an experimental test to check the validity of Eq. (10), or equivalently, time dilation is due to the Eq. (10). It shows that, in the absence of gravity, a clock at rest relative to another will run at the same rate, while one that moves relative to first will indicate its own time, different from the other. In particular, the velocity of motion in time $\frac{\partial t'}{\partial t}$ of S with respect to S´ depends on its spatial velocity v'.[4] In brief, time is the fourth dimension in which there is no freedom of motion because motion in time is not independent of motion in space, but depends on it as in Eq. (14). This dependence makes it possible to determine the spatial velocity of a fluid to go back in time. The answer will be found in the following sections.

2.3. Inertial field

The curvature tensor in S can be defined as a function of the affine connection

$$R^{\lambda}{}_{\mu\nu\sigma} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\lambda}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\sigma\alpha} - \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\lambda}_{\nu\alpha}. \tag{15}$$

Since the curvature tensor in S´ is clearly zero, it follows from the laws of transformation of the tensors that it is also zero in any other reference frame

$$R^{\lambda}{}_{\mu\nu\sigma} = 0. \tag{16}$$

This result implies that space-time in S is also flat instead of curved and that gravity does not appear either in the moving reference frame S.

The Ricci tensor is obtained by contraction of the curvature tensor and it is also zero according to Eq. (16)

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = 0. \tag{17}$$

This equation shows that the field in S has not sources but it is originated only by the motion of S with respect to S´, that is, the inertial field has not sources which curve space-time.

[4] As we know time also depends on the position in a gravitational field.

3. Rotating reference frames

There is no agreement either on how to establish the relativistic transformation to rotating frames. For this reason, we will use the same spatial coordinate transformations between the inertial and rotating frames as in [28] and [31-36] while time coordinate transformation will be determined by the gauge Eq. (10) compatible with time dilation Eq. (14).

3.1. Coordinate transformations

A rotating reference frame S is an accelerated reference frame with an interesting property: it moves in the same place.

In this case it is convenient to define cylindrical coordinates $x^{\mu} = (t, r, \theta, z)$ in S for practical reasons. If the motion is circular, two equations must be added

$$\left.\begin{aligned} x'^2 + y'^2 &= r^2 \\ z' &= z \end{aligned}\right\} \tag{18}$$

The coordinate transformations from the reference frame at rest S´ to the rotating reference frame S compatible with Eqs. (18) are

$$\left.\begin{aligned} t' &= t'(t, r) \\ x' &= r\cos[\theta - \varphi(t)] \\ y' &= r\,\mathrm{sen}[\theta - \varphi(t)] \\ z' &= z \end{aligned}\right\}. \tag{19}$$

These coordinate transformations can be obtained by solving the relativistic equations of motion (3) as we will show in Section 4.1.

The Jacobian of the transformation is

$$J(\mathbf{x}' \to \mathbf{x}) = r\frac{\partial t'}{\partial t}. \tag{20}$$

3.2. The metric tensor

The metric in S is calculated with Eqs. (5) and (19)

$$g_{tt} = -\left(\frac{\partial t'}{\partial t}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2 = -1, g_{tr} = -\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}, g_{t\theta} = -r^2\frac{d\varphi}{dt}, g_{tz} = 0,$$

$$g_{rr} = 1 - \left(\frac{\partial t'}{\partial r}\right)^2, g_{r\theta} = 0, g_{rz} = 0, \tag{21}$$

$$g_{\theta\theta} = r^2, g_{\theta z} = 0, g_{zz} = 1$$

where Eq. (10) has been used in the first.

The first of Eqs. (21) can be written as

$$\frac{\partial t'}{\partial t} = \pm\sqrt{1 + r^2\left(\frac{d\varphi}{dt}\right)^2}. \tag{22}$$

This equation has two solutions.[5] The positive solution implies that t' increases with t, at a rate compatible with time dilation as proved in section 2.2. The continuity of the metric (21) implies the positive sign in Eq. (22). However, the g_{tr} and $g_{t\theta}$ components of this metric may have a singularity that change their signs or only one of them, just like it happens at event horizon of a black hole with the radial component of the Schwarzschild metric [38], but in our case, there is a time singularity. The change of sign of the g_{tr} component makes possible the change of sign in Eq. (22) which ensures a decrease of t' as t increases and the travel to the past. This is another key point to travel to the past. In section 5.2 we will see if this is physically possible, which would allow the fluid to go back in time.

3.3. The Affine Connection

The affine connection is calculated through Eqs. (7), with the inverse of the metric tensor given by

$$g^{\mu\nu} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}\eta^{\alpha\beta}, \tag{23}$$

the metric Eqs. (21) and its derivatives

$$\Gamma^{t}_{tt} = \frac{\pm r\sqrt{1+r^2\left(\frac{d\varphi}{dt}\right)^2}\,\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^2 + r^2\frac{d\varphi}{dt}\frac{d^2\varphi}{dt^2}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2}, \Gamma^{t}_{tr} = \frac{r\left(\frac{d\varphi}{dt}\right)^2}{1+r^2\left(\frac{d\varphi}{dt}\right)^2},$$

$$\Gamma^{t}_{t\theta} = -\frac{r\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\frac{d\varphi}{dt}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2}, \Gamma^{t}_{rr} = \frac{\frac{\partial t'}{\partial t}\frac{\partial^2 t'}{\partial r^2}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2}, \Gamma^{t}_{\theta\theta} = \frac{r\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2},$$

$$\Gamma^{r}_{tt} = -r\left(\frac{d\varphi}{dt}\right)^2, \Gamma^{r}_{t\theta} = r\frac{d\varphi}{dt}, \Gamma^{r}_{\theta\theta} = -r, \Gamma^{\theta}_{tt} = \frac{\pm r\sqrt{1+r^2\left(\frac{d\varphi}{dt}\right)^2}\,\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^3 - \frac{d^2\varphi}{dt^2}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2} \tag{24}$$

$$\Gamma^{\theta}_{tr} = -\frac{\frac{d\varphi}{dt}}{r\left[1+r^2\left(\frac{d\varphi}{dt}\right)^2\right]}, \Gamma^{\theta}_{t\theta} = -\frac{r\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^2}{1+r^2\left(\frac{d\varphi}{dt}\right)^2}, \Gamma^{\theta}_{rr} = \frac{\frac{\partial t'}{\partial t}\frac{\partial^2 t'}{\partial r^2}\frac{d\varphi}{dt}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2},$$

[5] From now on, both signs will be shown except in the case when the choice of one of them is justified.

$$\Gamma^{\theta}_{r\theta} = \frac{1}{r}, \Gamma^{\theta}_{\theta\theta} = \frac{r\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\frac{d\varphi}{dt}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2}.$$

The rest of the components are zero.

4. Rotating perfect fluid without pressure

4.1. Equations of motion in the rotating reference frame

In this section we apply the above results to the simplest model of fluid, that is, a perfect fluid without pressure. A perfect fluid has no viscosity and no heat conduction in the Momentarily Comoving Reference Frame (MCRF). A pressure-free perfect fluid represents a fluid without interaction between particles, so there is no motion of the particles in the MCRF and neither its particles have random motion. It is like a model of dust. In this case the energy-momentum tensor of the perfect fluid is

$$T^{\mu\nu} = \rho \frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}. \tag{25}$$

where

$$d\tau^2 = -g_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{26}$$

is the equation of the invariant interval, being $d\tau$ the proper time.
If the fluid is at rest in the rotating reference frame S, then Eqs. (9), Eq. (10) with Eq. (26) and Eq. (25) give

$$\frac{dt}{d\tau} = 1 \tag{27}$$

and

$$T^{tt} = \rho, \quad T^{ti} = T^{it} = 0, \quad T^{ij} = 0. \tag{28}$$

In addition, if the mass-energy is conserved in S´ the continuity equation of the fluid must be[6]

$$f'^{t} = 0 \tag{29}$$

and since the components of the density of the external force f'^{μ} are calculated through Eqs. (4)

$$f'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} f^{\alpha} \tag{30}$$

the Eq. (29) gives

[6] This equation is not covariant so it is only valid in the reference frame at rest S´.

$$\rho = \rho_i \frac{1 + r^2\left(\frac{d\varphi}{dt}\right)^2_{t=t_i}}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2}. \tag{31}$$

This equation shows that the proper energy density of the fluid ρ is always positive or zero.

The components of the density of the external force applied to the fluid in S are calculated using Eqs. (3), (24), (28) and (31)[7]

$$f^t = \pm\rho_i r \frac{\left[1 + r^2\left(\frac{d\varphi}{dt}\right)^2_{t=t_i}\right]\sqrt{1 + r^2\left(\frac{d\varphi}{dt}\right)^2}\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^2}{\left[1 + r^2\left(\frac{d\varphi}{dt}\right)^2\right]^2}, \tag{32}$$

$$f^r = -\rho_i r \frac{\left[1 + r^2\left(\frac{d\varphi}{dt}\right)^2_{t=t_i}\right]\left(\frac{d\varphi}{dt}\right)^2}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2}, \tag{33}$$

$$f^\theta = \rho_i \frac{\left[1 + r^2\left(\frac{d\varphi}{dt}\right)^2_{t=t_i}\right]\left[\pm r\sqrt{1 + r^2\left(\frac{d\varphi}{dt}\right)^2}\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^3 - \frac{d^2\varphi}{dt^2}\right]}{\left[1 + r^2\left(\frac{d\varphi}{dt}\right)^2\right]^2}, \tag{34}$$

$$f^z = 0. \tag{35}$$

The rotational motion and the time travel are only due to the radial f^r and the tangential f^θ components of the density of the external force, Eqs. (33) and (34) and from now on, only these two components will be referred to.

Taking into account Eqs. (8) and introducing Eqs. (6) and Eqs. (25) into Eqs. (3) and adding the gauge given by Eqs. (9) and (10) and the constraints Eqs. (18), (26) and (29), we have a system of twelve differential equations with thirteen unknowns which leave one degree of freedom. The solutions are: the components of the four-velocity Eqs. (9) and (27), the coordinate transformations Eqs. (19) with Eq. (22), the proper energy density of the fluid Eq. (31) and the density of the external force Eqs. (32) to (35). They are solutions of the relativistic equations of motion in the rotating reference frame as a result of the chosen gauge, as expected by construction, and they depend on the only degree of freedom, that is, the angular velocity of the fluid. This guarantees that if we apply the density of the external force on the fluid given by Eqs. (33) to (35) in the rotating reference frame the coordinate transformations will be given by Eqs. (19) with Eq. (22) because they can be obtained by solving the relativistic equations of motion.

[7] The density of force in the S.I. is obtained dividing by c^2 taking $c \approx 3 \cdot 10^8 \, m/s$.

4.2. Equations of motion in the reference frame at rest

It is also convenient to determine the motion of the rotating fluid in the reference frame S´. The Eqs. (19) allow to obtain the density of the external force in S´ using Eqs. (29), (30), and Eqs. (32) to (35)

$$f'^t = 0, \tag{36}$$

$$f'^r = f^r, \tag{37}$$

$$f'^\theta = -\rho_i \frac{\left[1 + r^2 \left(\frac{d\varphi}{dt}\right)^2_{t=t_i}\right] \frac{d^2\varphi}{dt^2}}{\left[1 + r^2 \left(\frac{d\varphi}{dt}\right)^2\right]^2}, \tag{38}$$

$$f'^z = f^z = 0. \tag{39}$$

Also for the same reasons given for S, from now on only the radial f'^r and tangential f'^θ components of the density of the external force will be referred to.

In addition, introducing Eq. (22) into Eq. (13) gives the velocity of S

$$v' = \frac{r\frac{d\varphi}{dt}}{\sqrt{1 + r^2 \left(\frac{d\varphi}{dt}\right)^2}} \le 1. \tag{40}$$

If the angular velocity is constant $\frac{d\varphi}{dt} = \omega$ and the fluid is initially at rest at $t_i = t_0$ in S with density ρ_0, then the relativistic centripetal force given by Eqs. (37) and (33)[8]

$$f'^r_{rel} = -\rho_0 \frac{\omega^2 r}{1 + \omega^2 r^2} \tag{41}$$

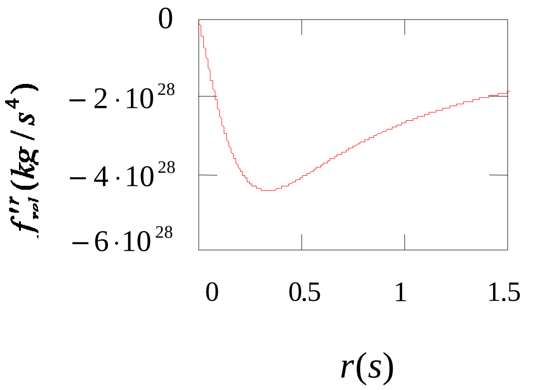

Fig. 1. f'^r_{rel} as function of r with constant angular velocity $\omega = 3\,rad/s$. The minimum is $f'^r_{rel} = -4.05\cdot10^{28}\,Kg/s^4 = -4.5\cdot10^{11}\,N/m^3$ at $r = 1/\omega$.

[8] ρ_0 can be chosen as the proper energy density of water, that is $\rho_0 = 27\cdot10^{27}\,Kg/s^3 \approx 9\cdot10^{19}\,J/m^3$.

is different from the classical law

$$f'^{\,r}_{clas} = -\rho_0 \omega^2 r. \tag{42}$$

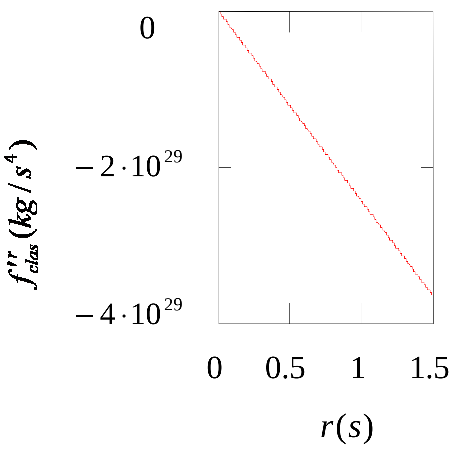

Fig. 2. Classical law for $\omega = 3\,rad/s$. $f'^{r}_{clas} = -8.1\cdot10^{28}\,kg/s^4 = -9\cdot10^{11}\,N/m^3$ at $r = 1/\omega$.

5. Time travel

In this section we apply the results obtained in the previous sections to determine the dynamics of the rotating perfect fluid without pressure to go back in time. This means finding the external force applied on the fluid to go back in time with rotational motion. For this purpose the only available degree of freedom of a rotating perfect fluid, that is, the angular velocity $\frac{d\varphi}{dt}$ is fixed. This variable is chosen so that the associated time obtained from Eq. (22) allows the fluid to go back in time. The travel will take five legs. The procedure is the following: first, we fix the angular velocity $\frac{d\varphi}{dt}$ and second, we calculate $\frac{\partial t'}{\partial t}$ through Eq. (22), integrating the last we obtain $t' = t'(t,r)$ and deriving this we calculate $\frac{\partial t'}{\partial r}$. In addition, we use Eq. (31). Then, we insert these results into the equations of motion (33) and (34). Occasionally, we will use Eqs. (37) and (38) to show the motion in S´.

5.1. Temporal acceleration

In general, due to the difficulty to synchronize accelerated clocks with inertial clocks, the clocks of both frames must be synchronized before starting the motion. At the beginning, the observers located in S´ and S are at rest in an inertial reference frame in which the Minkowski metric Eq. (1) is valid. This allows to synchronize the clocks of both frames according to the criterion of the light signals proposed by Einstein [39].

During the motion, moving clocks can be compared with clocks at rest except during the travel to the past as we saw in the introduction.
The first leg of time travel occurs during the interval $t_0 \leq t \leq t_1$ for an observer located in S. If the reference frame S is initially at rest at $t = t_0$ the simplest function $\frac{d\varphi}{dt}$ is

$$\frac{d\varphi_I}{dt} = a_1(t - t_0) \tag{43}$$

where a_1 is a constant that can be chosen so that

$$a_1 > 0. \tag{44}$$

In addition

$$\frac{\partial t'_I}{\partial t} = \sqrt{1 + a_1^2 r^2 (t - t_0)^2}, \tag{45}$$

$$t'_I = t'_0 + \frac{t - t_0}{2}\sqrt{1 + a_1^2 r^2 (t - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t - t_0) + \sqrt{1 + a_1^2 r^2 (t - t_0)^2}\right] \tag{46}$$

and

$$\frac{\partial t'_I}{\partial r} = \frac{t - t_0}{2r}\sqrt{1 + a_1^2 r^2 (t - t_0)^2} - \frac{1}{2a_1 r^2} Ln\left[a_1 r(t - t_0) + \sqrt{1 + a_1^2 r^2 (t - t_0)^2}\right] \geq 0. \tag{47}$$

We obtain for this fluid in the reference frame S during the first leg of the time travel

$$f_I^r = -\frac{\rho_0 a_1^2 r (t - t_0)^2}{1 + a_1^2 r^2 (t - t_0)^2}, \tag{48}$$

$$f_I^\theta = \rho_0 \frac{r\sqrt{1 + a_1^2 r^2 (t - t_0)^2}\,\frac{\partial t'_I}{\partial r} a_1^3 (t - t_0)^3 - a_1}{\left[1 + a_1^2 r^2 (t - t_0)^2\right]^2} \tag{49}$$

where $\frac{\partial t'_I}{\partial r}$ is given by Eq. (47).

5.2. Time reversal

The second leg of time travel occurs during the interval $t_1 \leq t \leq t_3$ for an observer located in S. The simplest function $\frac{d\varphi}{dt}$ which gives time reversal is

$$\frac{d\varphi_{II}}{dt} = \frac{1}{(t_2 - t)^{\frac{1}{3}}}\sqrt{a_2 (t_2 - t)^{\frac{2}{3}} + b_2}. \tag{50}$$

Introducing Eq. (50) into Eq. (22) we have

$$\frac{\partial t'_{II}}{\partial t} = \frac{1}{(t_2 - t)^{\frac{1}{3}}}\sqrt{(1 + a_2 r^2)(t_2 - t)^{\frac{2}{3}} + b_2 r^2}. \tag{51}$$

It is observed that $\frac{\partial t'_{II}}{\partial t}$ changes sign at $t = t_2$, which allows the fluid to go back in time as it was discussed in Eq. (22). Integrating Eq. (51) yields

$$t'_{II} = t'_1 + \frac{1}{1+a_2 r^2}\left[\left(1+a_2 r^2\right)\left(t_2 - t_1\right)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} - \frac{1}{1+a_2 r^2}\left[\left(1+a_2 r^2\right)\left(t_2 - t\right)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} \qquad (52)$$

where the function $t'_1 = t'_1(r)$ is calculated from the continuity of t'

$$t'_1 = t'_0 + \frac{t_1 - t_0}{2}\sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2}\right]. \qquad (53)$$

In order to avoid inconsistencies it is convenient that

$$a_2 > 0. \qquad (54)$$

The shape of the function given by Eq. (52) with Eq. (53) that reaches a maximum at $t = t_2$ allows time reversal which occurs at that time and also the travel to any time in the past as indicated in the introduction. Moreover, it is compatible with time dilation since it verifies Eq. (22).

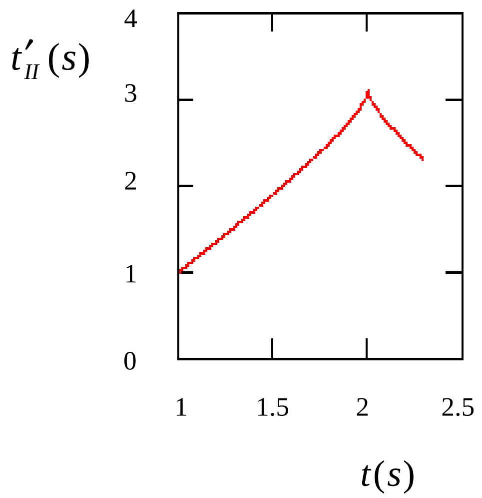

Fig 3. t'_{II} as function of t taking $r = r_0 = 10^{-7} s = 30m$ during the second leg of time travel. The constants can be chosen as: $a_1 = 1 rad/s^2$, $a_2 = 10^{14} rad^2/s^2$, $b_2 = 10^{14} rad^2/s^{4/3}$, $t_0 = 0$, $t_1 = 1s$, $t_2 = 2s$ and $t'_0 = 0$.

Deriving Eq. (52) and taking into account Eq. (53) we obtain

$$\frac{\partial t'_{II}}{\partial r} = \frac{t_1 - t_0}{2r}\sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2} - \frac{1}{2a_1 r^2} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2}\right]$$
$$+ \frac{a_2 r\left(1+a_2 r^2\right)\left(t_2 - t_1\right)^{\frac{2}{3}} + a_2 b_2 r^3 + 3b_2 r}{\left(1+a_2 r^2\right)^2}\sqrt{\left(1+a_2 r^2\right)\left(t_2 - t_1\right)^{\frac{2}{3}} + b_2 r^2}$$
$$- \frac{a_2 r\left(1+a_2 r^2\right)\left(t_2 - t\right)^{\frac{2}{3}} + a_2 b_2 r^3 + 3b_2 r}{\left(1+a_2 r^2\right)^2}\sqrt{\left(1+a_2 r^2\right)\left(t_2 - t\right)^{\frac{2}{3}} + b_2 r^2} > 0. \qquad (55)$$

The proper energy density at time $t = t_1$ is obtained from Eq. (31)

$$\rho_1 = \rho(t_1) = \frac{\rho_0}{1 + r^2\left(\frac{d\varphi_I}{dt}\right)^2_{t=t_1}}. \tag{56}$$

Introducing Eqs. (50) and (56) into Eqs. (33) and (34) we obtain for this fluid in the reference frame S during the second leg of the time travel

$$f_{II}^{r} = -\rho_0 r \frac{a_2 (t_2 - t)^{\frac{2}{3}} + b_2}{\left(1 + a_2 r^2\right)\left(t_2 - t\right)^{\frac{2}{3}} + b_2 r^2}, \tag{57}$$

$$f_{II}^{\theta} = \rho_0 \frac{r\sqrt{\left(1 + a_2 r^2\right)\left(t_2 - t\right)^{\frac{2}{3}} + b_2 r^2}\,\frac{\partial t'_{II}}{\partial r}\left[a_2 (t_2 - t)^{\frac{2}{3}} + b_2\right]^{\frac{3}{2}} - \frac{b_2}{3\sqrt{a_2 (t_2 - t)^{\frac{2}{3}} + b_2}}}{\left[\left(1 + a_2 r^2\right)\left(t_2 - t\right)^{\frac{2}{3}} + b_2 r^2\right]^2} \tag{58}$$

where $\frac{\partial t'_{II}}{\partial r}$ is given by Eq. (55).

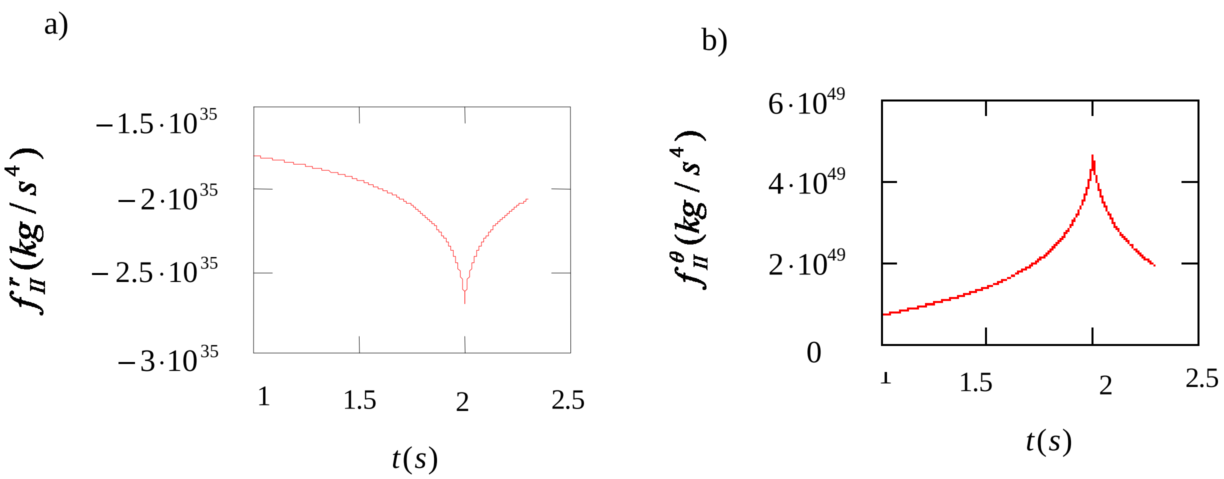

Fig 4. If we take $r = r_0 = 10^{-7}\,s = 30m$, $a_2 = 10^{14}\,rad^2/s^2$, $b_2 = 10^{14}\,rad^2/s^{4/3}$ and $\rho_0 = 27\cdot 10^{27}\,Kg/s^3 \approx 9\cdot 10^{19}\,J/m^3$ we obtain: a) f_{II}^{r} as function of t during the second leg of time travel. The minimum is $f_{II}^{r} = -2.7\cdot 10^{35}\,kg/s^4 = -3\cdot 10^{18}\,N/m^3$ at $t = t_2 = 2s$. b) f_{II}^{θ} as function of t during the second leg of time travel. The maximum is $f_{II}^{\theta} = 4.315\cdot 10^{49}\,kg/s^4$ $= 4.794\cdot 10^{32}\,N/m^3$ at $t = t_2 = 2s$.

This solution of the relativistic equations of motion makes the travel to the past physically possible for three reasons:

I) f_{II}^{r} and f_{II}^{θ} given by Eqs. (57) and (58) are finite through the interval, which ensures physical meaning.

II) The continuity of the metric in S given by Eqs. (21) implies the positive sign in Eq. (22). However, introducing Eqs. (50), (51) and (55) into Eqs. (22) we check that the metric in S is singular at $t = t_2$ while the metric in S´ is the Minkowski metric, free of singularities. At the singularity, the g_{tr} component of the metric Eqs. (21) may change the sign, just like it happens at event horizon of a black hole with the radial component of the Schwarzschild metric [38]. This change of sign makes possible the change of sign in Eq. (51) which ensures a decrease of t' as t increases and the travel to the past.

III) This change of sign is physically possible because the density of the external force applied on the fluid in S, Eq. (34) or (58) depends on the sign of Eq. (22), since $\frac{\partial t'_{II}}{\partial r} > 0$ and so an observer at rest on S can choose between one of them, applying the force to travel to the past.

Further, Eq. (51) explains the infinite redshift for the observer S´ of the wavelength of a photon emitted in S and also the similarities inside a black hole avoiding contradictions in the way indicated in the introduction.

It can also be verified that changing the sign of Eq. (58) also allows the travel to the past. Both solutions only differ in the sign of the angular velocity $\frac{d\varphi_{II}}{dt}$.

The second leg of time travel occurs during the interval $t'_1 \le t' \le t'_3$ for an observer located in S´.

The rotating motion of this fluid in this reference frame during this leg is

$$f'^{r}_{II} = f^{r}_{II} = -\rho_0 r \frac{a_2 (t_2 - t)^{\frac{2}{3}} + b_2}{\left(1 + a_2 r^2\right)(t_2 - t)^{\frac{2}{3}} + b_2 r^2}, \tag{59}$$

$$f'^{\theta}_{II} = -\frac{\rho_0 b_2}{3\sqrt{a_2 (t_2 - t)^{\frac{2}{3}} + b_2}\left[\left(1 + a_2 r^2\right)(t_2 - t)^{\frac{2}{3}} + b_2 r^2\right]^2}. \tag{60}$$

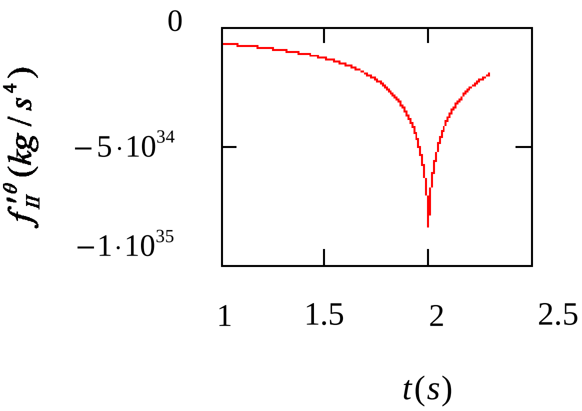

Fig. 5. f'^{θ}_{II} as function of t taking $r = r_0 = 10^{-7} s = 30m$, $a_2 = 10^{14} rad^2/s^2$, $b_2 = 10^{14} rad^2/s^{4/3}$ and $\rho_0 = 27\cdot 10^{27} Kg/s^3 \approx 9\cdot 10^{19} J/m^3$ during the second leg of time travel. The minimum is $f'^{\theta}_{II} = -9\cdot 10^{34} kg/s^4 = -10^{18} N/m^3$ at $t = t_2 = 2s$.

We also note that f'^{r}_{II} and f'^{θ}_{II} are finite through the interval.

It is observed that the density of the external force $f_{II}^{\prime\theta}$ given by Eq. (38) or (60) does not depend on $\frac{\partial t_{II}'}{\partial t}$. This means that an observer located in S´ cannot force the time reversal because the positive and negative solutions of Eq. (22) lead to the same $f_{II}^{\prime\theta}$. For this reason, $f_{II}^{\prime\theta}$ only provides $\frac{\partial t_{II}'}{\partial t} > 0$. On the other hand, an observer located in the reference frame S, physically distinguishes both solutions and he can choose the solution in Eq. (34) or (58) that causes the time reversal.

It can be advisable to write Eqs. (59) and (60) as a function of time t' measured in S´. For this, it is necessary to find t as a function of t' from Eq. (52).

It can be checked by inspection of Eqs. (40), (50) and (31) that the fluid reaches the speed of light in S´ at time $t = t_2$ because the proper energy density of the fluid is null at time in which time reversal occurs.

5.3. Back to the past

The third leg of time travel occurs during the interval $t_3 \le t \le t_4$ for an observer located in S. It is convenient to keep angular velocity $\frac{d\varphi}{dt}$ constant during this leg and thus

$$\frac{d\varphi_{III}}{dt} = a_3, \tag{61}$$

where a_3 is a constant so that

$$a_3 < 0. \tag{62}$$

In addition

$$\frac{\partial t_{III}'}{\partial t} = \sqrt{1 + a_3{}^2 r^2} \tag{63}$$

and

$$t_{III}' = t_3' + \sqrt{1 + a_3{}^2 r^2}\,(t - t_3) \tag{64}$$

where

$$t_3' = t_0' + \frac{t_1 - t_0}{2}\sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r (t_1 - t_0) + \sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2}\right]$$

$$+ \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} - \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}}. \tag{65}$$

Deriving Eq. (64) we obtain

$$\frac{\partial t'_{III}}{\partial r} = \frac{dt'_3}{dr} + \frac{a_3{}^2 r}{\sqrt{1+a_3{}^2 r^2}}(t-t_3). \tag{66}$$

It is necessary to determine the velocity Eq. (63) of the travel to the past. As $\frac{\partial t'_{III}}{\partial t}$ depends on r to determine this velocity it is convenient to fix r so that, in $r = r_0$ verifies

$$\left.\frac{\partial t'_{III}}{\partial t}\right|_{r_0, t_3} = -V \tag{67}$$

where V is a constant greater than zero

$$V > 0. \tag{68}$$

We obtain for this fluid in the reference frame S during the third leg of time travel

$$f^{\,r}_{III} = -\rho_0 r \frac{a_3{}^2}{1+a_3{}^2 r^2}, \tag{69}$$

$$f^{\,\theta}_{III} = \rho_0 \frac{r\sqrt{1+a_3{}^2 r^2}\,\frac{\partial t'_{III}}{\partial r} a_3{}^3}{\left[1++a_3{}^2 r^2\right]^2} \tag{70}$$

where $\frac{\partial t'_{III}}{\partial r}$ is given by Eqs. (65) and (66).

5.4. New time reversal

The fourth leg of time travel occurs during the interval $t_4 \le t \le t_6$ for an observer located in S. The simplest function $\frac{d\varphi}{dt}$ which gives a new time reversal is

$$\frac{d\varphi_{IV}}{dt} = \frac{1}{(t-t_5)^{\frac{1}{3}}}\sqrt{a_4 (t-t_5)^{\frac{2}{3}} + b_4}. \tag{71}$$

where t_5, a_4 and b_4 are constants verifying

$$t_4 < t_5 < t_6, b_4 > 0, b_4 > -a_4 (t_4 - t_5)^{\frac{2}{3}}. \tag{72}$$

Introducing Eq. (71) into Eq. (22) we have

$$\frac{\partial t'_{IV}}{\partial t} = \frac{1}{(t-t_5)^{\frac{1}{3}}}\sqrt{(1+a_4 r^2)(t-t_5)^{\frac{2}{3}} + b_4 r^2}. \tag{73}$$

It is observed that $\frac{\partial t'_{IV}}{\partial t}$ changes sign at $t = t_5$ as it was discussed in Eq. (22).

Integrating Eq. (73) yields

$$t'_{IV} = t'_4 - \frac{1}{1+a_4 r^2}\left[(1+a_4 r^2)(t_4 - t_5)^{\frac{2}{3}} + b_4 r^2\right]^{\frac{3}{2}} + \frac{1}{1+a_4 r^2}\left[(1+a_4 r^2)(t - t_5)^{\frac{2}{3}} + b_4 r^2\right]^{\frac{3}{2}} \tag{74}$$

where

$$t'_4 = t'_0 + \frac{t_1 - t_0}{2}\sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2}\right]$$

$$- \frac{1}{1 + a_2 r^2}\left[\left(1 + a_2 r^2\right)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} + \frac{1}{1 + a_2 r^2}\left[\left(1 + a_2 r^2\right)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}}$$

$$+ \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{\left(1 + a_2 r^2\right)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2}\,(t_4 - t_3). \quad (75)$$

In order to avoid inconsistencies in Eq. (74) it is convenient that

$$a_4 > 0. \quad (76)$$

The shape of the function Eq. (74) that reaches a minimum at $t = t_5$ allows a new time reversal which occurs at that time.

Deriving Eq. (74) we obtain

$$\frac{\partial t'_{IV}}{\partial r} = \frac{dt'_4}{dr} - \frac{a_4 r\left(1 + a_4 r^2\right)(t_4 - t_5)^{\frac{2}{3}} + a_4 b_4 r^3 + 3b_4 r}{\left(1 + a_4 r^2\right)^2}\sqrt{\left(1 + a_4 r^2\right)(t_4 - t_5)^{\frac{2}{3}} + b_4 r^2}$$

$$+ \frac{a_4 r\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + a_4 b_4 r^3 + 3b_4 r}{\left(1 + a_4 r^2\right)^2}\sqrt{\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2} < 0. \quad (77)$$

The proper energy density at time $t = t_4$ is calculated from Eq. (31)

$$\rho_4 = \rho(t_4) = \frac{\rho_0}{1 + r^2\left(\frac{d\varphi_{III}}{dt}\right)^2_{t=t_4}}. \quad (78)$$

Introducing Eqs. (71) and (78) into Eqs. (33) and (34) we obtain for this fluid in the reference frame S during the fourth leg of time travel

$$f^r_{IV} = -\rho_0 r \frac{a_4 (t - t_5)^{\frac{2}{3}} + b_4}{\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2}, \quad (79)$$

$$f^\theta_{IV} = \rho_0 \frac{r\sqrt{\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2}\,\frac{\partial t'_{IV}}{\partial r}\left[a_4 (t - t_5)^{\frac{2}{3}} + b_4\right]^{\frac{3}{2}} + \frac{b_4}{3\sqrt{a_4 (t - t_5)^{\frac{2}{3}} + b_4}}}{\left[\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2\right]^2} \quad (80)$$

where $\frac{\partial t'_{IV}}{\partial r}$ is given by Eqs. (75) and (77). We also note that f^r_{IV} and f^θ_{IV} are finite through the interval although the metric in Eq. (21) is singular at $t = t_5$.

Again, it is observed that the new time reversal is possible because f^θ_{IV} depends on the sign of $\frac{\partial t'_{IV}}{\partial t}$ because $\frac{\partial t'_{IV}}{\partial r} < 0$.

We can prove that $f_{IV}'^{r}$ and $f_{IV}'^{\theta}$ are finite through the interval. Again, it can be verified that at time $t = t_5$ in S, the fluid reaches the speed of light in S´.

5.5. Temporal deceleration

The fifth leg of time travel occurs during the interval $t_6 \le t \le T$ for an observer located in S. If the rotation stops at the end of the travel, at time $t = T$ in S, the simplest function $\frac{d\varphi}{dt}$ is

$$\frac{d\varphi_V}{dt} = a_5 (t - T) \tag{81}$$

where a_5 is a constant such that

$$a_5 < 0. \tag{82}$$

In addition

$$\frac{\partial t_V'}{\partial t} = \sqrt{1 + a_5^{\,2} r^2 (t - T)^2} \tag{83}$$

and

$$t_V' = t_6' - \frac{t_6 - T}{2}\sqrt{1 + a_5^{\,2} r^2 (t_6 - T)^2} - \frac{1}{2a_5 r} Ln\left[a_5 r(t_6 - T) + \sqrt{1 + a_5^{\,2} r^2 (t_6 - T)^2}\right]$$
$$+ \frac{t - T}{2}\sqrt{1 + a_5^{\,2} r^2 (t - T)^2} + \frac{1}{2a_5 r} Ln\left[a_5 r(t - T) + \sqrt{1 + a_5^{\,2} r^2 (t - T)^2}\right]. \tag{84}$$

where

$$t_6' = t_0' + \frac{t_1 - t_0}{2}\sqrt{1 + a_1^{\,2} r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1^{\,2} r^2 (t_1 - t_0)^2}\right]$$
$$- \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} + \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}}$$
$$+ \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2}\,(t_4 - t_3) + \frac{1}{1 + a_4 r^2}\left[(1 + a_4 r^2)(t_6 - t_5)^{\frac{2}{3}} + b_4 r^2\right]^{\frac{3}{2}}$$
$$- \frac{1}{1 + a_4 r^2}\left[(1 + a_4 r^2)(t_4 - t_5)^{\frac{2}{3}} + b_4 r^2\right]^{\frac{3}{2}}. \tag{85}$$

It is convenient that the travel ends at a certain time T' in the past of S´. As $t' = t'(t, r)$, it is advisable to fix r in which this happens. This will be $r = r_0$ and so

$$t_V'(T)_{r_0} = T' \tag{86}$$

Deriving Eq. (84) we obtain

$$\frac{\partial t_V'}{\partial r} = \frac{dt_6'}{dr} - \frac{t_6 - T}{2r}\sqrt{1 + a_5^{\,2} r^2 (t_6 - T)^2} + \frac{1}{2a_5 r^2} Ln\left[a_5 r(t_6 - T) + \sqrt{1 + a_5^{\,2} r^2 (t_6 - T)^2}\right]$$

$$+\frac{t-T}{2r}\sqrt{1+a_5{}^2r^2(t-T)^2}-\frac{1}{2a_5r^2}Ln\left[a_5r(t-T)+\sqrt{1+a_5{}^2r^2(t-T)^2}\right]. \tag{87}$$

We obtain for this fluid in the reference frame S during the fifth leg of time travel

$$f_V^{\,r}=-\frac{\rho_0a_5{}^2r(t-T)^2}{1+a_5{}^2r^2(t-T)^2}, \tag{88}$$

$$f_V^{\,\theta}=\rho_0\frac{r\sqrt{1+a_5{}^2r^2(t-T)^2}\,\frac{\partial t'_V}{\partial r}\,a_5{}^3(t-T)^3-a_5}{\left[1+a_5{}^2r^2(t-T)^2\right]^2} \tag{89}$$

where $\frac{\partial t'_V}{\partial r}$ is given by Eqs. (85) and (87). Observe that the constants appearing in the equations have been chosen to avoid inconsistencies and they will be constrained by Eqs. (67), (86) and continuity conditions. Free constants can be chosen to minimize the external force applied to achieve time reversal.

And this is the end of the travel to the past. As it is easy to imagine, the return travel to the present can be carried out in a similar way in only three legs that do not entail as much difficulty as the travel to the past.

Finally, note that the travel to the past can be used in interstellar space travel. Indeed, due to the effect of time dilation, if a traveller reached a speed close to that of light he could reach his destination star in a few months. However, when he returned to Earth he will note that here hundreds or thousands of years have passed. If the traveler went back in time he could make the duration of the travel the same for both the traveller and people on Earth.

Appendix A: Einstein's equations with scalar field

The above results are adequate in a flat space-time in the absence of gravity when the gravitational field created by the fluid is so weak that it barely curves space-time which happens in practice. Although it is not relevant in the exposed developments, it is convenient to generalize the formalism to the case in which the gravitational field generated by the fluid is important. The results in the absence of gravity are obtained as a particular case. To do this, it is necessary to resort to Einstein's field equations.

First, it should be noted that the energy-momentum tensor Eqs. (25) is not conserved, as it is deduced from Eqs. (3), so that it is not compatible with Einstein's equations

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=-8\pi GT^{\mu\nu} \tag{A1}$$

for which the energy-momentum tensor is conserved. This means that the energy-momentum tensor Eqs. (25) is not adequate to calculate the gravitational field created by a perfect fluid subjected to mechanical forces.

The simplest way to avoid this difficulty is introducing a scalar field ϕ into Einstein's Eqs. (A1) similarly to the Brans-Dicke Theory [40]. This scalar field verifies a new differential equation. The simplest generally covariant field equation for such a scalar field is

$$\phi_{;\ ;\alpha}^{\ \alpha} = 4\pi\lambda T_{M\alpha}^{\alpha} \tag{A2}$$

where λ is a coupling constant and $T_M^{\mu\nu}$ is the energy-momentum tensor of the matter when gravity is present.

The most general symmetric tensor for this field must contain terms which involves two derivatives of one or two ϕ fields

$$T_{\phi}^{\mu\nu} = A(x)\phi_{;}^{\ \mu}\phi_{;}^{\ \nu} + B(x)g^{\mu\nu}\phi_{;}^{\ \alpha}\phi_{;\alpha} + C(x)\phi_{;\ ;}^{\ \mu\ \nu} + D(x)g^{\mu\nu}\phi_{;\ ;\alpha}^{\ \alpha}. \tag{A3}$$

Einstein's Eqs. (A1) including this scalar field are

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi G\left(T_M^{\ \mu\nu} + T_{\phi}^{\mu\nu}\right) \tag{A4}$$

where $T_M^{\mu\nu}$ verifies

$$f^{\mu} = T_{M\ ;\alpha}^{\mu\alpha} \tag{A5}$$

and $T_{\phi}^{\mu\nu}$ is the energy-momentum tensor of the scalar field given by Eqs. (A3).

In order to determine the functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$ which appear in Eqs. (A3), the covariant divergence of Eqs. (A4) is calculated obtaining

$$\begin{aligned} f^{\mu} = &-\frac{\partial A}{\partial x^{\alpha}}\phi_{;}^{\mu}\phi_{;}^{\alpha} - \left[A(x) + 2B(x)\right]\phi_{;\ ;}^{\ \mu\ \alpha}\phi_{;\alpha} - A(x)\phi_{;}^{\mu}\phi_{;\ ;\alpha}^{\alpha} - \frac{\partial B}{\partial x^{\beta}}g^{\mu\beta}\phi_{;}^{\alpha}\phi_{;\alpha} \\ &-\frac{\partial C}{\partial x^{\alpha}}\phi_{;\ ;}^{\mu\ \alpha} - C(x)\phi_{;\alpha;\ ;}^{\ \ \alpha\ \mu} - \frac{\partial D}{\partial x^{\beta}}g^{\mu\beta}\phi_{;\ ;\alpha}^{\alpha} - D(x)\phi_{;\ ;\ ;\alpha}^{\mu\ \alpha} \end{aligned} \tag{A6}$$

where Eqs. (A3) and (A5) have been used. These are the equations of the dynamics of the scalar field that determine the four functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$ if the density of the external forces f^{μ} are known.

A.1. Reference frames

The ten Eqs. (A4) of the gravitational field are not independent but are related by the four Bianchi identities which reduces Eqs. (A4) to six independent equations. To determine an unambiguous metric, it is necessary to add four more equations to fix the gauge, that is, the reference frame.

In the presence of gravity, it is also convenient to define a reference frame at rest S´ as described in section 2. Since Eqs. (3) are the same as in a Gravitational field in the presence of external forces [30], [37] according to the Principle of Equivalence, then in any reference frame, for example, the reference frame S´, the equations of motion of the fluid will be

$$f'^{\mu} = T'^{\mu\alpha}_{M\ ;\alpha} = \frac{\partial T'^{\mu\alpha}_{M}}{\partial x'^{\alpha}} + \Gamma'^{\alpha}_{\alpha\beta}T'^{\mu\beta}_{M} + \Gamma'^{\mu}_{\alpha\beta}T'^{\alpha\beta}_{M}. \tag{A7}$$

Now, Eqs. (2) must be also verified in the reference frame at rest S´

$$f'^{\mu} = \frac{\partial T_M'^{\mu\alpha}}{\partial x'^{\alpha}}. \qquad \text{(A8)}$$

This is equivalent, in view of Eqs. (A7) to choose a gauge such that

$$\Gamma'^{\alpha}_{\alpha\beta} T_M'^{\mu\beta} + \Gamma'^{\mu}_{\alpha\beta} T_M'^{\alpha\beta} = 0. \qquad \text{(A9)}$$

These four equations that determine the gauge, that is, the reference frame S´, are not covariant so they are not valid in any reference frame, but only in S´.

The moving reference frame S is chosen, as in section 2, such that the fluid, which originates the gravitational field, is at rest[9]

$$\frac{dx^i}{d\tau} = 0. \qquad \text{(A10)}$$

In addition, to fix completely the reference frame S, it is necessary to add another condition that it can be obtained as in Eq. (10) by imposing that the time-time component of the metric must be the same in any reference frame

$$g_{tt} = g'_{tt} \qquad \text{(A11)}$$

where g'_{tt} is the time-time component of the metric in the reference frame S´ at rest.

The three Eqs. (A10) together with Eq. (A11) fix another gauge, that is, the reference frame S.

Once the reference frames have been fixed and the metric is known in S´, as a solution of Einstein's Eqs. (A4) in the gauge Eqs. (A9), we can proceed as in the previous sections.

A.2. Relevance of the scalar field

The introduction of the scalar field ϕ gives the Minkowski metric Eqs. (1) as a solution of Einstein's Eqs. (A4) in the gauge Eqs. (A9). This solution is adequate in the absence of gravity. In practice, this happens when the gravitational field created by the source is so weak that it barely curves space-time. In this situation, in any reference frame, it is verified

$$R^{\mu\nu} = 0 \qquad \text{(A12)}$$

and

$$R = g_{\alpha\beta} R^{\alpha\beta} = 0. \qquad \text{(A13)}$$

It can be shown from Eqs. (A12), (A13) and (A4)

$$T_M^{\mu\nu} = -T_\phi^{\mu\nu}, \qquad \text{(A14)}$$

so the ϕ field is especially important.

In vacuum, the energy-momentum tensor of the matter is zero

$$T_M^{\mu\nu} = 0 \qquad \text{(A15)}$$

and Eq. (A2) will be written as

[9] It is possible to find a reference frame like this because external forces can apply in such a way that all the particles of the fluid are at rest.

$$\phi_{;\ ;\alpha}^{\alpha} = 0 \quad \text{(A16)}$$

while Eqs. (A5) give

$$f^{\mu} = 0. \quad \text{(A17)}$$

Introducing Eqs. (A16) and (A17) into Eqs. (A6) we obtain

$$0 = \frac{\partial A}{\partial x^{\alpha}} \phi_{;}^{\mu} \phi_{;}^{\alpha} + \left[A(x) + 2B(x)\right] \phi_{;\ ;}^{\mu\ \alpha} \phi_{;\alpha} + \frac{\partial B}{\partial x^{\beta}} g^{\mu\beta} \phi_{;}^{\alpha} \phi_{;\alpha} + \frac{\partial C}{\partial x^{\alpha}} \phi_{;\ ;}^{\mu\ \alpha} + C(x) \phi_{;\alpha;\ ;}^{\ \ \ \alpha\ \mu}. \quad \text{(A18)}$$

One solution of these equations, only valid on the outside of a static source, is

$$A(x) = B(x) = C(x) = 0. \quad \text{(A19)}$$

From Eqs. (A16) and (A19) the energy-momentum tensor Eqs. (A3) of the scalar field ϕ in vacuum is

$$T_{\phi}^{\mu\nu} = 0. \quad \text{(A20)}$$

Equations (A15) and (A20) imply that in this case Eqs. (A1) and (A4) have the same solutions in vacuum and the scalar field does not change the solutions of the Eqs. (A1) that explain, among other phenomena, the dynamics of the planets. In other cases, their significance will have to be determined.

A.3. Cosmological Dark Energy

The scalar field may be considered as the source of dark energy [41-44]. The introduction of the scalar field ϕ into Einstein's Eqs. (A4) through energy-momentum tensor Eqs. (A3) adds four independent functions to the field equations. If the mass-energy of the matter is conserved

$$T_{M\ \ ;\alpha}^{\ t\alpha} = 0 \quad \text{(A21)}$$

only three degrees of freedom remain. This has important cosmological implications because three of the four functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$, of Eqs. (A3), are independent and they can be adjusted to the observational data of accelerated expansion of the Universe.

Conclusions

The exposed developments demonstrate the theoretical possibility of travelling to the past, in the case of a macroscopic perfect fluid without pressure in the absence of gravity and without violating the laws of Special Relativity. This is only possible if the fluid is accelerated to the speed of light. At this time, the metric has a singularity that makes possible the time reversal, arguing that this is physically possible. To show all this, a relativistic treatment of rotation has been made using the principle of General Covariance, which has proved to be practical. Finally, it is necessary to introduce a Scalar Field into Einstein's equations to explain relativistic dynamics satisfactorily. This Scalar Field is the origin of the inertia and it has made possible to generalize the formalism to the case in which the fluid generates an appreciable Gravitational Field.

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