The Physics of Time Travel

In memory of my father who lives

Auhtor: Javier Sánchez Sánchez

THE PHYSICS OF TIME TRAVEL

Abstract

The principle of General Covariance allows us to understand the dynamics of accelerated reference frames using the formalism of General Relativity. Using this principle, the metric, the affine connection and the equations of the dynamics in an accelerated reference frame, in the absence of gravity, are deduced easily. The equations obtained explain: the geometry of a space-time without gravity, the differences between gravity and inertia, what is time and how it is possible to travel in time. These results are applied to the dynamics of a simple system: a perfect fluid without pressure and in circular motion. In this case there are some indications about the possible origin of dark matter and dark energy. Finally, the external force applied to this rotating fluid is calculated to go back in time. In the Appendix the formalism is generalized to the case in which gravity is present.

INTRODUCTION

The invariance of the equations of Physics under Lorentz transformations forms one of the pillars of Physics. This principle determines the shape of the equations of Physics in any inertial frame.

It makes sense that there is a principle of invariance more general that lets equations of Physics be express in any accelerated frame. This is the principle of invariance of the equations of Physics under any coordinate transformations which is implicit in the principle of General Covariance. This principle is an alternative version of the Principle of Equivalence.

In general, the invariance under Lorentz transformations is satisfactory when the mass does not appear in the equations of the field, as in electromagnetism. If mass is present, as in the case of mechanics or gravity, then general invariance is more appropriate.

The principle of General Covariance is highly adequate to find the equations of dynamics in accelerated reference frames, the metric, and the affine connection in a space-time without gravity. Its main advantage is that the coordinate transformations between observers in motion appear explicit in the equations of dynamics. This results lets the dynamics of the motion be calculated from a known coordinate transformations. Another important advantage comes from the Principle of the Equivalence of Gravitation and Inertia postulated by Einstein in the General Relativity. As we know, it is easier to handle inertia appearing with the accelerated motion than gravity, although the effects in both cases must be locally equivalent.

In this paper the treatment of moving reference frames is shown from a non-quantum but relativistic point of view and using the formalism of General Relativity.

Firstly, sections I to IV consider the motion in space-time without gravity. Section I studies the moving reference frames and the field. Section II, analyses the dynamics in a rotating reference frame. In section III, the previous results are applied to study the motion of the rotating perfect fluid without pressure. In section IV, the force applied on this model of rotating fluid is calculated to go back in time. Finally, the Appendix connects with the equations of the field and gravity.

I. MOVING REFERENCE FRAMES

Equations of motion

When relativistic dynamics is studied, it is convenient to define two observers linked to two different reference frames: one that is considered at rest and the other moving with respect to the first. From now on they are going to be known as reference frame at rest S´ and moving reference frame S.

Let S´ be an inertial reference frame in which the laws of Special Relativity are valid globally. In this frame the metric is the Minkowski metric[1]

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta_{\nu\mu} \tag{1.1}$$

An observer located in S´ defines Cartesian coordinates $x'^{\mu} = (t', x', y', z')$ to measure events.
Let S be a moving reference frame with respect to S´ in which other coordinates $x^{\mu} = (x^0, x^1, x^2, x^3)$ are used to measure the events.
The equations of motion in S´ of a fluid moving with respect to S´ under the action of an external force are

$$f'^{\mu} = \frac{\partial T'^{\mu\alpha}}{\partial x'^{\alpha}} \tag{1.2}$$

where

$$T'^{\mu\nu} = p\eta^{\mu\nu} + (p+\rho)\frac{dx'^{\mu}}{d\tau}\frac{dx'^{\nu}}{d\tau} \tag{1.3}$$

is the energy-momentum tensor of a perfect fluid, f'^{μ} is the density of the external force,

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta^{\nu\mu} \tag{1.4}$$

is the inverse of the metric (1.1), that is,

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[1]From now on, any index that appears twice, once as a subscript and once as a superscript is understood to be summed over; and a system of units in which the speed of light is unity ($c = 1$) is used.

$$\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^{\mu}{}_{\nu} \tag{1.5}$$

$\dfrac{dx'^{\mu}}{d\tau}$ is the velocity four-vector, p and ρ are the pressure and the proper energy density and $d\tau$ is the proper time verifying

$$d\tau^2 = -\eta_{\alpha\beta}dx'^{\alpha}dx'^{\beta} \tag{1.6}$$

The equations of motion must be invariant under any coordinate transformations.
Suppose that the coordinate transformations between S′ and S are continuous functions of the form

$$x'^{\mu} = x'^{\mu}(x^0, x^1, x^2, x^3) \tag{1.7}$$

These coordinate transformations must be global, covering all space-time or at least the volume of the fluid.
The inverse transformations are functions of the form

$$x^{\mu} = x^{\mu}(t', x', y', z') \tag{1.8}$$

Then, carrying out any coordinate transformations from S′ to S, such as (1.7) in the equations of motion (1.2), we can find the equations of motion of a fluid in any reference frame, for example S

$$f^{\mu} = T^{\mu\alpha}{}_{;\alpha} = \frac{\partial T^{\mu\alpha}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\alpha\beta}T^{\mu\beta} + \Gamma^{\mu}_{\alpha\beta}T^{\alpha\beta} \tag{1.9}$$

where

$$f^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} f'^{\alpha} \tag{1.10}$$

is the density of the external force acting on the fluid,

$$T^{\mu\nu} = pg^{\mu\nu} + (p+\rho)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} \tag{1.11}$$

is the energy-momentum tensor of a perfect fluid and $\dfrac{dx^{\mu}}{d\tau}$ is the velocity four-vector measured in S.
Also, from Equation (1.6) it follows

$$d\tau^2 = -\eta_{\alpha\beta}dx'^{\alpha}dx'^{\beta} = -\eta_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial x'^{\beta}}{\partial x^{\gamma}}dx^{\delta}dx^{\gamma} = -g_{\delta\gamma}dx^{\delta}dx^{\gamma} \tag{1.12}$$

where

$$g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}} = g_{\nu\mu} \tag{1.13}$$

is the metric in S.
The affine connection is defined as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}}\frac{\partial^2 x'^{\alpha}}{\partial x^{\mu}\partial x^{\nu}} = \Gamma^{\lambda}_{\nu\mu} \tag{1.14}$$

In addition, $g^{\mu\nu}$ is the inverse tensor of the metric (1.13), that is

$$g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}{}_{\nu} \tag{1.15}$$

Equations (1.9) are invariant under any coordinate transformations. They explicitly show the coordinate transformations (1.7) through the affine connection (1.14).

Degrees of freedom

Equations (1.9) together with condition (1.12) form a system of five equations with 14 unknowns: the four components of velocity four-vector $\frac{dx^{\mu}}{d\tau}$, the proper energy density ρ, the pressure p, the four components of the density of the external force f^{μ}, and the four functions that relate the coordinates in both reference frames $x'^{\mu} = x'^{\mu}(\boldsymbol{x})$, which leave 14-5=9 degrees of freedom. The equation of state

$$p = p(\rho) \tag{1.16}$$

provides another equation that reduces the degrees of freedom to eight.

If the reference frame S is chosen so that the fluid remains at rest with respect to it, then

$$\frac{dx^{i}}{d\tau} = 0 \tag{1.17}$$

and three more equations are obtained.
Equations (1.17) do not fix the Gauge alone, that is, the reference frame. It is necessary to add another condition: the time-time component of the metric has to be invariant[(1)]

$$g_{tt} = \eta_{tt} = -1 \tag{1.18}$$

where one of the Equations (1.1) has been used.
The three Equations (1.17) plus Equation (1.18) reduce the degrees of freedom to four.

In addition, the continuity equation of the fluid must be verified. When the mass-energy is conserved in S´, it takes the form

$$f'^{t} = \frac{\partial T'^{t\alpha}}{\partial x'^{\alpha}} = 0 \tag{1.19}$$

and this reduces the degrees of freedom to three, as corresponds to a perfect fluid in motion.

The same applies to S´: the four Equations (1.2), together with condition (1.6), the equation of state (1.16) and the continuity equation (1.19) form a system of seven equations with ten unknowns, which allows only three degrees of freedom. These three degrees of freedom correspond to the three spatial components of the density of the external force acting on the fluid that can be chosen arbitrarily. The rest of variables will be functions of them.
However, it is possible to choose any three functions that appear in the dynamics as independent variables, for example, the functions that relate the spatial coordinates in both reference frames and the rest will be functions of them.

Meaning of time

As it is well known, the three spatial dimensions are characterized by freedom of motion. This is not so in the case of time as it is demonstrated below.

Introducing Equations (1.13) into Equation (1.18) gives

[(1)]This condition is needed for both the fluid and the reference frame S comove in time.

$$g_{tt} = -\left(\frac{\partial t'}{\partial t}\right)^2 + \left(\frac{\partial x'}{\partial t}\right)^2 + \left(\frac{\partial y'}{\partial t}\right)^2 + \left(\frac{\partial z'}{\partial t}\right)^2 = -1 \tag{1.20}$$

or

$$\left(\frac{\partial x'}{\partial t}\right)^2 + \left(\frac{\partial y'}{\partial t}\right)^2 + \left(\frac{\partial z'}{\partial t}\right)^2 = -1 + \left(\frac{\partial t'}{\partial t}\right)^2 \tag{1.21}$$

On the other hand, the components of the velocity of S with respect to S′ are

$$v'^i = \frac{dx'^i}{dt'} = \frac{\dfrac{\partial x'^i}{\partial x^\alpha} dx^\alpha}{\dfrac{\partial t'}{\partial x^\beta} dx^\beta} = \frac{\dfrac{\partial x'^i}{\partial t} dt + \dfrac{\partial x'^i}{\partial x^j} dx^j}{\dfrac{\partial t'}{\partial t} dt + \dfrac{\partial t'}{\partial x^k} dx^k} = \frac{\dfrac{\partial x'^i}{\partial t} + \dfrac{\partial x'^i}{\partial x^j} v^j}{\dfrac{\partial t'}{\partial t} + \dfrac{\partial t'}{\partial x^k} v^k} = \frac{\dfrac{\partial x'^i}{\partial t}}{\dfrac{\partial t'}{\partial t}} \tag{1.22}$$

where

$$v^j = \frac{dx^j}{dt} = 0 \tag{1.23}$$

since S is at rest in its own reference frame.
In addition, the magnitude of the velocity can be calculated with Equations (1.22) and (1.21):

$$v' = \sqrt{v'^i v'^i} = \sqrt{\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 + \left(\frac{dz'}{dt'}\right)^2} = \sqrt{\frac{\left(\frac{\partial x'}{\partial t}\right)^2 + \left(\frac{\partial y'}{\partial t}\right)^2 + \left(\frac{\partial z'}{\partial t}\right)^2}{\left(\frac{\partial t'}{\partial t}\right)^2}}$$

$$= \sqrt{\frac{-1 + \left(\frac{\partial t'}{\partial t}\right)^2}{\left(\frac{\partial t'}{\partial t}\right)^2}} \le 1 \tag{1.24}$$

This equation shows, as expected, that it can never be greater than unity, that is, any reference frame can't move at a speed greater than the speed of light, with respect to S′.

From Equation (1.24), it is found

$$\frac{\partial t'}{\partial t} = \frac{1}{\sqrt{1 - v'^2}} = \gamma' \tag{1.25}$$

This equation explains time dilation that occurs when velocity increases. It shows that, in the absence of gravity, a clock at rest relative to another will run at the same rate, while one that moves relative to first will indicate its own time, different from the other. It shows that motion in time depends on motion in space. In particular, the velocity of motion in time $\frac{\partial t'}{\partial t}$ of S with respect to S′ depends on its spatial velocity v'.
Consequently, time is a dimension in which there is no freedom of motion because motion in time is not independent of motion in space, but depends on it as in Equation (1.25).This dependence, which is explicit in the Equations (1.9) in S through the affine connection (1.14), makes it possible to determine the spatial velocity of a fluid to go back in time. The answer will be found in the following sections.

Inertial Field

The affine connection can be calculated using Equations (1.14) or the metric $g_{\mu\nu}$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} + \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right) \tag{1.26}$$

The curvature tensor in S can be defined as a function of the affine connection

$$R^{\lambda}{}_{\mu\nu\sigma} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\lambda}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\sigma\alpha} - \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\lambda}_{\nu\alpha} \tag{1.27}$$

Since the curvature tensor in S´ is clearly zero, it follows from the laws of transformation of the tensors that it is also zero in any other reference frame

$$R^{\lambda}{}_{\mu\nu\sigma} = 0 \tag{1.28}$$

This result implies that space-time in S is also flat instead of curved and that gravity does not appear either in the moving reference frame S.
The Ricci tensor is obtained by contraction of the curvature tensor and it is also zero according to Equation (1.28)

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = 0 \tag{1.29}$$

This equation shows that the field in S has no sources but it is originated only by the motion of S with respect to S´.
In brief, the inertial field has not sources which curve space-time.

II. ROTATING REFERENCE FRAMES

Coordinate transformations

A rotating reference frame S is an accelerated reference frame with an interesting property: it moves in the same place.

In this case it is convenient to define cylindrical coordinates $x^{\mu} = (t, r, \theta, z)$ in S for practical reasons. If the motion is circular, it must add two equations to Equations (1.9) and (1.12):

$$\left.\begin{aligned} x'^2 + y'^2 &= r^2 \\ z' &= z \end{aligned}\right\} \tag{2.1}$$

which reduces to one the degrees of freedom of a perfect fluid at rest with respect to S.

The coordinate transformations from the rotating reference frame S to the reference frame at rest S´ (1.7) compatible with Equations (2.1) are functions

$$\left.\begin{aligned} t' &= t'(t, r) \\ x' &= r cos\left[\theta - \varphi(t)\right] \\ y' &= r sen\left[\theta - \varphi(t)\right] \\ z' &= z \end{aligned}\right\} \tag{2.2}$$

The metric tensor

The metric in S can be calculated with Equations (1.13), (1.1) and (2.2):

$$g_{tt} = -\left(\frac{\partial t'}{\partial t}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2 = -1$$
$$g_{tr} = -\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r} = g_{rt}$$
$$g_{t\theta} = -r^2\frac{d\varphi}{dt} = g_{\theta t}$$
$$g_{tz} = 0 = g_{zt}$$
$$g_{rr} = 1 - \left(\frac{\partial t'}{\partial r}\right)^2 \tag{2.3}$$
$$g_{r\theta} = 0 = g_{\theta r}$$
$$g_{rz} = 0 = g_{zr}$$
$$g_{\theta\theta} = r^2$$
$$g_{\theta z} = 0 = g_{z\theta}$$
$$g_{zz} = 1$$

where Equation (1.18) has been used in the first.
The determinant of the metric tensor is

$$det\ g_{\mu\nu} = g_{tt}g_{rr}g_{\theta\theta} - g_{tr}{}^2 g_{\theta\theta} - g_{rr}g_{t\theta}{}^2 = -r^2\left[1 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] = -g \tag{2.4}$$

The first of Equations (2.3) can be written as

$$\frac{\partial t'}{\partial t} = \pm\sqrt{1 + r^2\left(\frac{d\varphi}{dt}\right)^2} \tag{2.5}$$

This equation has two solutions[(1)]. The positive solution implies that t' increases with t, which is known as time dilation. In section IV it will be seen when negative solution is admissible, which ensures a decrease of t' as t increases, what it would allow back in time.

Inverse of the metric tensor

The inverse of the metric can be calculated with Equations (1.15) and (2.3):

$$g^{tt} = \frac{-1 + \left(\frac{\partial t'}{\partial r}\right)^2}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2}$$

$$g^{tr} = \frac{-\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2} = g^{rt}$$

[(1)] From now on, both signs will be shown except when one of them is justified.

$$g^{t\theta} = -\frac{\left[1-\left(\frac{\partial t'}{\partial r}\right)^2\right]\frac{d\varphi}{dt}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2} = g^{\theta t}$$

$$g^{tz} = 0 = g^{zt}$$

$$g^{rr} = 1 \qquad (2.6)$$

$$g^{r\theta} = -\frac{\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\frac{d\varphi}{dt}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2} = g^{\theta r}$$

$$g^{rz} = 0 = g^{zr}$$

$$g^{\theta\theta} = \frac{1+r^2\left(\frac{\partial t'}{\partial r}\right)^2\left(\frac{d\varphi}{dt}\right)^2}{r^2\left[1+r^2\left(\frac{d\varphi}{dt}\right)^2\right]}$$

$$g^{\theta z} = 0 = g^{z\theta}$$

$$g^{zz} = 1$$

The affine connection

Deriving Equation (2.5) we obtain

$$\frac{\partial^2 t'}{\partial t^2} = \frac{r^2\frac{d\varphi}{dt}\frac{d^2\varphi}{dt^2}}{\pm\sqrt{1+r^2\left(\frac{d\varphi}{dt}\right)^2}}; \qquad \frac{\partial^2 t'}{\partial t\partial r} = \frac{\partial^2 t'}{\partial r\partial t} = \frac{r\left(\frac{d\varphi}{dt}\right)^2}{\pm\sqrt{1+r^2\left(\frac{d\varphi}{dt}\right)^2}} \qquad (2.7)$$

The affine connection can be calculated with Equation (1.26), the derivatives of the metric (2.3), the inverse of the metric tensor (2.6) and Equations (2.7):

$$\Gamma^t_{tt} = \frac{\pm r\sqrt{1+r^2\left(\frac{d\varphi}{dt}\right)^2}\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^2 + r^2\frac{d\varphi}{dt}\frac{d^2\varphi}{dt^2}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2}$$

$$\Gamma^r_{tt} = -r\left(\frac{d\varphi}{dt}\right)^2$$

$$\Gamma^\theta_{tt} = \frac{\pm r\sqrt{1+r^2\left(\frac{d\varphi}{dt}\right)^2}\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^3 - \frac{d^2\varphi}{dt^2}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2}$$

$$\Gamma^{t}_{rt} = \frac{r\left(\frac{d\varphi}{dt}\right)^2}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2} = \Gamma^{t}_{tr}$$

$$\Gamma^{t}_{\theta t} = -\frac{r\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\frac{d\varphi}{dt}}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2} = \Gamma^{t}_{t\theta}$$

$$\Gamma^{t}_{rr} = \frac{\frac{\partial t'}{\partial t}\frac{\partial^2 t'}{\partial r^2}}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2}$$

$$\Gamma^{t}_{\theta\theta} = \frac{r\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2} \tag{2.8}$$

$$\Gamma^{r}_{t\theta} = r\frac{d\varphi}{dt} = \Gamma^{r}_{\theta t}$$

$$\Gamma^{\theta}_{tr} = -\frac{\frac{d\varphi}{dt}}{r\left[1 + r^2\left(\frac{d\varphi}{dt}\right)^2\right]} = \Gamma^{\theta}_{rt}$$

$$\Gamma^{\theta}_{t\theta} = -\frac{r\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^2}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2} = \Gamma^{\theta}_{\theta t}$$

$$\Gamma^{r}_{\theta\theta} = -r$$

$$\Gamma^{\theta}_{rr} = \frac{\frac{\partial t'}{\partial t}\frac{\partial^2 t'}{\partial r^2}\frac{d\varphi}{dt}}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2}$$

$$\Gamma^{\theta}_{r\theta} = \frac{1}{r} = \Gamma^{\theta}_{\theta r}$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{r\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}\frac{d\varphi}{dt}}{1 + r^2\left(\frac{d\varphi}{dt}\right)^2}$$

The rest of the components are zero.

III. ROTATING PERFECT FLUID WITHOUT PRESSURE

Equations of motion in the rotating reference frame

In this section we apply the above results to the simplest model of fluid, that is, a perfect fluid without pressure. In this case Equation (1.16) is replaced by

$$p = 0 \tag{3.1}$$

The energy-momentum tensor of the perfect fluid in S is obtained by introducing Equation (3.1) into Equations (1.11)

$$T^{\mu\nu} = \rho \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.2}$$

This model represents a fluid without interaction between particles, so it is like a model of dust.

If the fluid is at rest in the rotating reference frame S, then Equations (1.17) are right, and using cylindrical coordinates they take the form

$$\frac{dr}{d\tau} = 0; \quad \frac{d\theta}{d\tau} = 0; \quad \frac{dz}{d\tau} = 0 \tag{3.3}$$

These equations together with Equations (1.12) and (1.18) imply that

$$\frac{dt}{d\tau} = 1 \tag{3.4}$$

In this reference frame the energy-momentum tensor is obtained with Equations (3.2), (3.3) and (3.4)

$$T^{tt} = \rho; \quad T^{ti} = T^{it} = 0; \quad T^{ij} = 0 \tag{3.5}$$

The components of the density of the external force applied to the fluid in S are calculated using Equations (1.9), (2.8) and (3.5)

$$f^t = \frac{\partial\rho}{\partial t} + 2\rho\Gamma^t_{tt} + \rho\Gamma^\theta_{\theta t} = \frac{\partial\rho}{\partial t} + \rho \frac{\pm r\sqrt{1+r^2\left(\frac{d\varphi}{dt}\right)^2}\,\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^2}{1+r^2\left(\frac{d\varphi}{dt}\right)^2} + 2\rho\frac{r^2\frac{d\varphi}{dt}\frac{d^2\varphi}{dt^2}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2} \tag{3.6}$$

$$f^r = \rho\Gamma^r_{tt} = -\rho\, r\left(\frac{d\varphi}{dt}\right)^2 \tag{3.7}$$

$$f^\theta = \rho\Gamma^\theta_{tt} = \rho\frac{\pm r\sqrt{1+r^2\left(\frac{d\varphi}{dt}\right)^2}\,\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^3 - \frac{d^2\varphi}{dt^2}}{1+r^2\left(\frac{d\varphi}{dt}\right)^2} \tag{3.8}$$

$$f^z = 0 \tag{3.9}$$

The components of the density of the external force f'^μ in S′ can be calculated through Equations (1.10)

$$f'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} f^\alpha \tag{3.10}$$

In particular, the time component of Equation (3.10) can be calculated with the first one of Equations (2.2) and Equations (3.6) and (3.7)

$$f'^{t}=\frac{\partial t'}{\partial t}f^{t}+\frac{\partial t'}{\partial r}f^{r}=\frac{\partial t'}{\partial t}\frac{\partial \rho}{\partial t}+\frac{\partial t'}{\partial t}\rho\frac{\partial}{\partial t}\left\{Ln\left[1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}\right]\right\} \tag{3.11}$$

The conservation of mass-energy in S′ requires that Equation (1.19) is verified and together with Equation (3.11) give

$$\frac{\partial \rho}{\rho}=-\partial Ln\left[1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}\right] \tag{3.12}$$

Integrating the last equation between t_i and t, the proper energy density of the fluid ρ is calculated

$$\rho=\rho(t_i)\frac{1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}_{t=t_i}}{1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}}=\rho_i\frac{1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}_{t=t_i}}{1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}} \tag{3.13}$$

Deriving Equation (3.13) we obtain

$$\frac{\partial \rho}{\partial t}=-2\rho_i r^{2}\frac{\left[1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}_{t=t_i}\right]\frac{d\varphi}{dt}\frac{d^{2}\varphi}{dt^{2}}}{\left[1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}\right]^{2}} \tag{3.14}$$

Finally, the equations of motion (3.6) to (3.9) taking into account Equations (3.13) and (3.14) are

$$f^{t}=\pm\rho_i r\frac{\left[1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}_{t=t_i}\right]\sqrt{1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}}\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^{2}}{\left[1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}\right]^{2}} \tag{3.15}$$

$$f^{r}=-\rho_i r\frac{\left[1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}_{t=t_i}\right]\left(\frac{d\varphi}{dt}\right)^{2}}{1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}} \tag{3.16}$$

$$f^{\theta}=\rho_i\frac{\left[1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}_{t=t_i}\right]\left[\pm r\sqrt{1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}}\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^{3}-\frac{d^{2}\varphi}{dt^{2}}\right]}{\left[1+r^{2}\left(\frac{d\varphi}{dt}\right)^{2}\right]^{2}} \tag{3.17}$$

$$f^{z}=0 \tag{3.18}$$

As seen at the beginning of Section II, a rotating perfect fluid has only one degree of freedom. By inspecting Equations (3.15) to (3.18) we see that this degree of freedom is the angular velocity of the rotation $\frac{d\varphi}{dt}$ of S′ with respect to S, which can be chosen as an independent variable. The rest of variables will be functions of it. However, although there is only one degree of freedom, the density of the external force applied f^{μ} has two components: the radial f^{r} and the tangential f^{θ} calculated in Equations (3.16) and (3.17). Both components must be applied to produce circular motion. From now on, only these two components will be referred to.

Equations of motion in the reference frame at rest

It is also convenient to determine the motion of the rotating fluid in the reference frame S´. This can be done, in Cartesian coordinates, through Equations (3.10), (2.2) and (3.15) to (3.18). From a practical point of view it is also convenient to use cylindrical coordinates in S´. The relation between the cylindrical coordinates in S´ and S is obtained through Equations (2.2)

$$\left.\begin{aligned} & t' = t'(t,r) \\ & r' = \sqrt{x'^2 + y'^2} = r \\ & \theta' = arctg\,\frac{y'}{x'} = \theta - \varphi(t) \\ & z' = z \end{aligned}\right\} \tag{3.19}$$

This relationship allows to obtain the density of the external force directly in S´ in cylindrical coordinates using Equations (1.19), (3.10) and (3.15) to (3.19)

$$f'^{t} = 0 \tag{3.20}$$

$$f'^{r} = f^{r} \tag{3.21}$$

$$f'^{\theta} = -\frac{d\varphi}{dt} f^{t} + f^{\theta} = -\rho_i \frac{\left[1 + r^2\left(\dfrac{d\varphi}{dt}\right)^2_{t=t_i}\right]\dfrac{d^2\varphi}{dt^2}}{\left[1 + r^2\left(\dfrac{d\varphi}{dt}\right)^2\right]^2} \tag{3.22}$$

$$f'^{z} = f^{z} = 0 \tag{3.23}$$

The same result would have been obtained if first f'^{μ} was calculated in S´ in Cartesian coordinates and then performing a transformation to the cylindrical coordinates in S´ through Equations (3.10). Also for the same reasons given for S, from now on only the radial f'^{r} and tangential f'^{θ} components of the density of the external force will be referred to.

Classical Limit

The classical limit of equations of motion (3.21) and (3.22) is obtained at low velocities. Introducing Equation (2.5) into Equation (1.24) we obtain

$$v' = \frac{r\dfrac{d\varphi}{dt}}{\sqrt{1 + r^2\left(\dfrac{d\varphi}{dt}\right)^2}} \le 1 \tag{3.24}$$

It follows that $v' \ll 1$ when

$$r\frac{d\varphi}{dt} \ll 1 \tag{3.25}$$

Taking into account Equations (3.25) and (3.16) the classical limit of Equations (3.21) and (3.22) are

$$f'^{r} \approx -\rho_0 r\left(\frac{d\varphi}{dt}\right)^2 \tag{3.26}$$

$$f'^{\theta} \approx -\rho_0 \frac{d^2\varphi}{dt^2} \tag{3.27}$$

where for simplicity it has been assumed that the fluid is initially at rest at time $t_i = t_0$ in S in which the density is ρ_0. It is convenient to express f'^{r} and f'^{θ} as a function of time t' in S´. For this purpose, Equation (2.5) is solved with the approximation (3.25)

$$t' - t'_i \approx t - t_i \tag{3.28}$$

Finally, Equations (3.26) and (3.27) with Equation (3.28) provide

$$f'^{r} \approx -\rho_0 r\left(\frac{d\varphi}{dt'}\right)^2 \tag{3.29}$$

$$f'^{\theta} \approx -\rho_0 \frac{d^2\varphi}{dt'^2} \tag{3.30}$$

which is the classical result for a rotating fluid without pressure.

Dark matter

If the angular velocity is constant

$$\frac{d\varphi}{dt} = \omega \tag{3.31}$$

and the fluid is initially at rest at $t_i = t_0$ in S in which the density is ρ_0, then the relativistic centripetal force (3.21) with Equation (3.16),

$$f'^{\,r}_{rel} = -\rho_0 r\frac{\left(\frac{d\varphi}{dt}\right)^2}{1+r^2\left(\frac{d\varphi}{dt}\right)^2} = -\rho_0\frac{\omega^2 r}{1+\omega^2 r^2} \tag{3.32}$$

and the classical centripetal force (3.26)

$$f'^{r}_{clas} = -\rho_0\omega^2 r \tag{3.33}$$

do not depend on time and $f'^{r}_{rel} < f'^{r}_{clas}$. That implies

$$\rho_{rel} = \frac{\rho_{clas}}{1+\omega^2 r^2} \tag{3.34}$$

This means that in the classical case at constant angular velocity, it will be necessary a higher proper energy density than in the relativistic one. The interpretation is obvious: to explain the motion of the rotating fluid it is necessary dark matter in the classical case while it is not necessary with relativistic formalism.

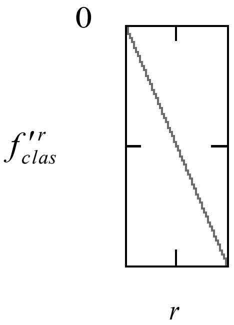

Fig. 1. Plotting f''^{r}_{clas} versus r at constant angular velocity ω.

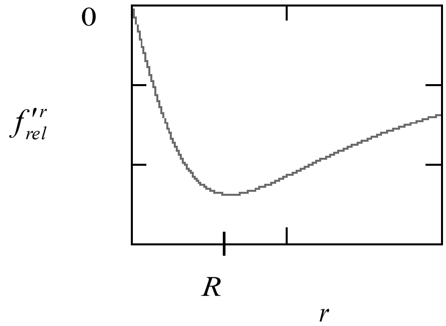

Fig. 2. Plotting f''^{r}_{rel} versus r at constant angular velocity ω.

The discrepancy between one outcome and another will be more evident for large v'. Introducing Equation (3.31) into Equation (3.24)

$$v' = \frac{\omega r}{\sqrt{1+\omega^2 r^2}} \tag{3.35}$$

it is observed that v' increases with r. This means that near the centre of rotation v' is non-relativistic, whereas far from the centre of rotation v' becomes relativistic, as it corresponds to a rotating motion. Specifically, for low angular velocities ω, v' will approximate the unit for large r. On the contrary, if ω is large v' will be relativistic even for small r.

The greatest negative value of f''^{r}_{rel} is when

$$\frac{df'_{rel}}{dr} = 0 \tag{3.36}$$

which occurs, taking into account Equation (3.32) in

$$R = \frac{1}{\omega} \tag{3.37}$$

Dark energy

The first term of Equation (3.17)

$$\pm \rho_i r \frac{\left[1+r^2\left(\frac{d\varphi}{dt}\right)^2_{t=t_i}\right]\sqrt{1+r^2\left(\frac{d\varphi}{dt}\right)^2}\,\frac{\partial t'}{\partial r}\left(\frac{d\varphi}{dt}\right)^3}{\left[1+r^2\left(\frac{d\varphi}{dt}\right)^2\right]^2} \tag{3.38}$$

is a term of dark energy. It is a relativistic consequence due to the dependence of t' with r. When $\frac{\partial t'}{\partial t}$ is positive, it must be

$$\frac{\partial t'}{\partial r} > 0 \tag{3.39}$$

Then, if the angular acceleration $\frac{d^2\varphi}{dt^2}$ and angular velocity $\frac{d\varphi}{dt}$ have the same sign, which indicates positive or negative acceleration, then the inertial force f^{θ} given by

(3.17) decreases due to the dark energy. On the contrary if they have different sign, a positive or negative deceleration occurs and the inertial force f^{θ} increases.
For low angular velocity $\frac{d\varphi}{dt}$, this effect will be important for large r. If angular velocity $\frac{d\varphi}{dt}$ is large, the effect will be noticeable for small r.

Note that according to Equation (3.22) the dark energy does not appear in the reference frame at rest S´ but it is exclusive in the rotating reference frame S.
The mass-energy is not conserved in S due to the dark energy as it can be deduced from Equation (3.15).

IV. TIME TRAVEL

In this section we apply the results obtained in the previous sections to determine the dynamics of the rotating perfect fluid without pressure to be back in time. This means finding the external force applied on the fluid to be back in time with the rotational motion. For this purpose the only degree of freedom of a rotating perfect fluid is fixed. It is convenient to choose, among all the variables, the angular velocity $\frac{d\varphi}{dt}$ that appears in the equations of motion (3.16) and (3.17) as independent variable. This variable is chosen so that the associated time shift obtained from Equation (2.5) allows the fluid to be back in time. The travel will take five legs.

Temporal Acceleration

At the beginning, the observers located in S´ and S are in an inertial reference frame in which the metric (1.1) is valid. This allows synchronizing the clocks of both frames according to the criterion of the light signals proposed by Einstein [1].
The first leg of time travel occurs during the interval $t_0 \le t \le t_1$ for an observer located in S. If the reference frame S is initially at rest at $t = t_0$ the simplest function $\varphi(t)$ is

$$\varphi_I(t) = \frac{a_1}{2}(t - t_0)^2 \tag{4.1}$$

The angular velocity is

$$\frac{d\varphi_I}{dt} = a_1(t - t_0) \tag{4.2}$$

which is initially zero

$$\left.\frac{d\varphi_I}{dt}\right|_{t=t_0} = 0 \tag{4.3}$$

The angular acceleration is

$$\frac{d^2\varphi_I}{dt^2} = a_1 \tag{4.4}$$

where a_1 is a constant that can be chosen so that

$$a_1 > 0 \tag{4.5}$$

Introducing Equation (4.2) into Equation (2.5) we obtain

$$\frac{\partial t_I'}{\partial t} = \sqrt{1 + r^2\left(\frac{d\varphi_I}{dt}\right)^2} = \sqrt{1 + a_1{}^2 r^2 (t - t_0)^2} \tag{4.6}$$

The function $t_I' = t_I'(t, r)$ is obtained by integration

$$t_I' = t_0' + \int_{t_0}^{t} \sqrt{1 + a_1{}^2 r^2 (t - t_0)^2}\, dt = t_0' + \frac{t - t_0}{2}\sqrt{1 + a_1{}^2 r^2 (t - t_0)^2} + \frac{1}{2 a_1 r} Ln\left[a_1 r (t - t_0) + \sqrt{1 + a_1{}^2 r^2 (t - t_0)^2}\right] \tag{4.7}$$

Deriving Equation (4.7)

$$\frac{\partial t_I'}{\partial r} = \frac{t - t_0}{2r}\sqrt{1 + a_1{}^2 r^2 (t - t_0)^2} - \frac{1}{2 a_1 r^2} Ln\left[a_1 r (t - t_0) + \sqrt{1 + a_1{}^2 r^2 (t - t_0)^2}\right] \geq 0 \tag{4.8}$$

Introducing Equations (4.2) to (4.4) into Equations (3.16) and (3.17) we obtain for this fluid in the reference frame S during the first leg of the time travel

$$f_I^r = -\rho_0 r \frac{\left(\frac{d\varphi_I}{dt}\right)^2}{1 + r^2\left(\frac{d\varphi_I}{dt}\right)^2} = -\frac{\rho_0 a_1{}^2 r (t - t_0)^2}{1 + a_1{}^2 r^2 (t - t_0)^2} \tag{4.9}$$

$$f_I^\theta = \rho_0 \frac{r\sqrt{1 + r^2\left(\frac{d\varphi_I}{dt}\right)^2}\frac{\partial t_I'}{\partial r}\left(\frac{d\varphi_I}{dt}\right)^3 - \frac{d^2\varphi_I}{dt^2}}{\left[1 + r^2\left(\frac{d\varphi_I}{dt}\right)^2\right]^2} = \rho_0 \frac{r\sqrt{1 + a_1{}^2 r^2 (t - t_0)^2}\frac{\partial t_I'}{\partial r} a_1{}^3 (t - t_0)^3 - a_1}{\left[1 + a_1{}^2 r^2 (t - t_0)^2\right]^2} \tag{4.10}$$

where $\frac{\partial t_I'}{\partial r}$ is given by Equation (4.8).

The first leg of time travel occurs during the interval $t_0' \leq t' \leq t_1'$ for an observer located in S´. The rotating motion of this fluid in this reference frame during this leg is obtained by introducing Equations (4.2) and (4.4) with Equation (3.28) into Equations (3.29) and (3.30)

$$f_I'^r \approx -\rho_0 a_1{}^2 r (t' - t_0')^2 \tag{4.11}$$

$$f_I'^\theta = -\rho_0 a_1 \tag{4.12}$$

These expressions are valid if condition (3.25) is verified, which taking into account Equation (4.2) it takes the form

$$r\frac{d\varphi_I}{dt} = a_1 r (t - t_0) < a_1 r (t_1 - t_0) << 1 \tag{4.13}$$

Time reversal

The second leg of time travel occurs during the interval $t_1 \leq t \leq t_3$ for an observer located in S. The simplest function $\varphi(t)$, which gives time reversal is

$$\varphi_{II}(t) = \varphi_{II}(t_1) - \frac{1}{a_2}\left[a_2 (t_2 - t)^{\frac{2}{3}} + b_2\right]^{\frac{3}{2}} + \frac{1}{a_2}\left[a_2 (t_2 - t_1)^{\frac{2}{3}} + b_2\right]^{\frac{3}{2}} \tag{4.14}$$

where t_2, a_2 y b_2 are constants checking

$$t_1 < t_2 < t_3;\ \ b_2 > 0;\ \ b_2 > -a_2(t_2 - t_1)^{\frac{2}{3}} \tag{4.15}$$

The angular velocity is

$$\frac{d\varphi_{II}}{dt} = \frac{1}{(t_2 - t)^{\frac{1}{3}}}\sqrt{a_2(t_2 - t)^{\frac{2}{3}} + b_2} \tag{4.16}$$

The angular acceleration is

$$\frac{d^2\varphi_{II}}{dt^2} = \frac{b_2}{3(t_2 - t)^{\frac{4}{3}}\sqrt{a_2(t_2 - t)^{\frac{2}{3}} + b_2}} \tag{4.17}$$

The functions $\varphi(t)$ and $\frac{d\varphi}{dt}$ must be continuous at $t = t_1$

$$\varphi_I(t_1) = \varphi_{II}(t_1) \tag{4.18}$$

This equation together with Equations (4.1) and (4.14) provides

$$\varphi_{II}(t_1) = \frac{a_1}{2}(t_1 - t_0)^2 \tag{4.19}$$

and

$$\left.\frac{d\varphi_I}{dt}\right|_{t=t_1} = \left.\frac{d\varphi_{II}}{dt}\right|_{t=t_1} \tag{4.20}$$

The last equation with Equations (4.2) and (4.16) leads to

$$a_1 = \frac{1}{t_1 - t_0}\sqrt{a_2 + \frac{b_2}{(t_2 - t_1)^{\frac{2}{3}}}} \tag{4.21}$$

Introducing Equation (4.16) into Equation (2.5) gives

$$\frac{\partial t'_{II}}{\partial t} = \pm\sqrt{1 + r^2\left(\frac{d\varphi_{II}}{dt}\right)^2} = \frac{1}{(t_2 - t)^{\frac{1}{3}}}\sqrt{(1 + a_2 r^2)(t_2 - t)^{\frac{2}{3}} + b_2 r^2} \tag{4.22}$$

It is observed that $\frac{\partial t'_{II}}{\partial t}$ changes sign at $t = t_2$ as it was discussed in Equation (2.5).

Integrating Equation (4.22) yields

$$t'_{II} = t'_1 + \int_{t_1}^{t} \frac{1}{(t_2 - t)^{\frac{1}{3}}}\sqrt{(1 + a_2 r^2)(t_2 - t)^{\frac{2}{3}} + b_2 r^2}\, dt$$

$$= t'_1 - \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} + \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} \tag{4.23}$$

Continuity of t' implies

$$t'_{II}(t_1) = t'_I(t_1) \tag{4.24}$$

that with Equation (4.23) gives

$$t'_1 = t'_I(t_1) \tag{4.25}$$

Introducing Equation (4.7) at $t = t_1$ into Equation (4.25)

$$t'_1 = t'_0 + \frac{t_1 - t_0}{2}\sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2}\right] \tag{4.26}$$

Inserting Equation (4.26) into Equation (4.23)

$$t'_{II} = t'_0 + \frac{t_1 - t_0}{2}\sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r (t_1 - t_0) + \sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2}\right]$$

$$- \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} + \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} \qquad (4.27)$$

In order to avoid divergences in Equation (4.27) it is convenient that

$$a_2 > 0 \qquad (4.28)$$

The form of the function (4.27) that reaches a maximum at $t = t_2$ allows time reversal. Time reversal occurs at time $t = t_2$ in S. This is only possible if the density of external force given by Equations (3.16) and (3.17) applied on this fluid in S and (3.21) and (3.22) in S´ are finite.

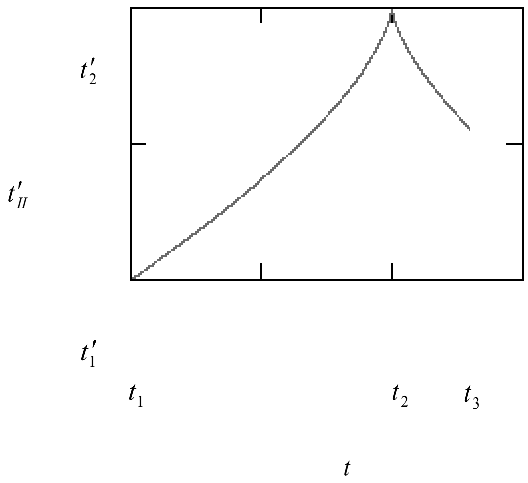

Fig. 3. Plotting t'_{II} versus t with fixed r during the second leg of time travel.

Deriving Equation (4.23), making use of Equations (4.25) and (4.8) at $t = t_1$ we obtain

$$\frac{\partial t'_{II}}{\partial r} = \frac{t_1 - t_0}{2r}\sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2} - \frac{1}{2a_1 r^2} Ln\left[a_1 r (t_1 - t_0) + \sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2}\right]$$

$$- \frac{a_2 r (1 + a_2 r^2)(t_2 - t)^{\frac{2}{3}} + a_2 b_2 r^3 + 3 b_2 r}{(1 + a_2 r^2)^2}\sqrt{(1 + a_2 r^2)(t_2 - t)^{\frac{2}{3}} + b_2 r^2}$$

$$+ \frac{a_2 r (1 + a_2 r^2)(t_2 - t_1)^{\frac{2}{3}} + a_2 b_2 r^3 + 3 b_2 r}{(1 + a_2 r^2)^2}\sqrt{(1 + a_2 r^2)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2} \geq 0 \qquad (4.29)$$

The proper energy density at time $t = t_1$ is obtained from Equations (3.13) and (4.3)

$$\rho_1 = \rho(t_1) = \frac{\rho_0}{1 + r^2 \left(\frac{d\varphi_I}{dt}\right)^2_{t = t_1}} \qquad (4.30)$$

Introducing Equations (4.16), (4.17) and (4.30) into Equations (3.16) and (3.17) and making use of Equation (4.20) we obtain for this fluid in the reference frame S during the second leg of the time travel

$$f_{II}^{r} = -\rho_0 r \frac{\left(\dfrac{d\varphi_{II}}{dt}\right)^2}{1 + r^2\left(\dfrac{d\varphi_{II}}{dt}\right)^2} = -\rho_0 r \frac{a_2 (t_2 - t)^{\frac{2}{3}} + b_2}{\left(1 + a_2 r^2\right)(t_2 - t)^{\frac{2}{3}} + b_2 r^2} \tag{4.31}$$

$$f_{II}^{\theta} = \rho_0 \frac{\pm r\sqrt{1 + r^2\left(\dfrac{d\varphi_{II}}{dt}\right)^2}\dfrac{\partial t'_{II}}{\partial r}\left(\dfrac{d\varphi_{II}}{dt}\right)^3 - \dfrac{d^2\varphi_{II}}{dt^2}}{\left[1 + r^2\left(\dfrac{d\varphi_{II}}{dt}\right)^2\right]^2}$$

$$= \rho_0 \frac{r\sqrt{\left(1 + a_2 r^2\right)(t_2 - t)^{\frac{2}{3}} + b_2 r^2}\dfrac{\partial t'_{II}}{\partial r}\left[a_2 (t_2 - t)^{\frac{2}{3}} + b_2\right]^{\frac{3}{2}} - \dfrac{b_2}{3\sqrt{a_2 (t_2 - t)^{\frac{2}{3}} + b_2}}}{\left[\left(1 + a_2 r^2\right)(t_2 - t)^{\frac{2}{3}} + b_2 r^2\right]^2} \tag{4.32}$$

where $\dfrac{\partial t'_{II}}{\partial r}$ is given by Equation (4.29). It is observed that f_{II}^{r} and f_{II}^{θ} are finite through the interval as we wanted.

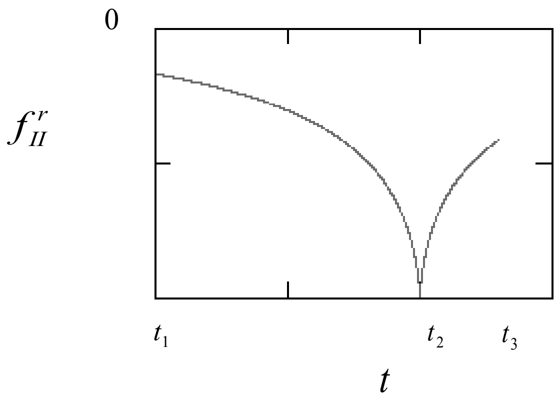

Fig. 4. Plotting f_{II}^{r} versus t with fixed r during the second leg of time travel.

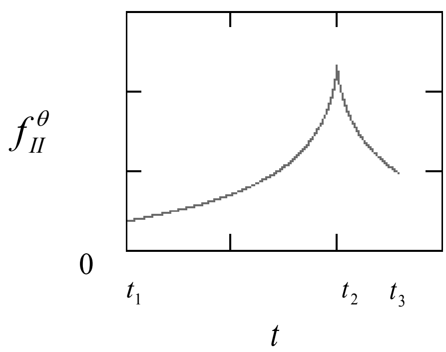

Fig. 5. Plotting f_{II}^{θ} versus t with fixed r during the second leg of time travel.

The second leg of time travel occurs during the interval $t'_1 \le t' \le t'_3$ for an observer located in S´. The rotating motion of this fluid in this reference frame during this leg is obtained with Equations (3.21) and (4.31), introducing Equations (4.16), (4.17) and (4.30) into Equation (3.22) and taking into account Equations (4.3) and (4.20)

$$f_{II}^{\prime r} = f_{II}^{r} = -\rho_0 r \frac{a_2 (t_2 - t)^{\frac{2}{3}} + b_2}{\left(1 + a_2 r^2\right)(t_2 - t)^{\frac{2}{3}} + b_2 r^2} \tag{4.33}$$

$$f_{II}'^{\theta} = -\rho_0 \frac{\dfrac{d^2\varphi_{II}}{dt^2}}{\left[1+r^2\left(\dfrac{d\varphi_{II}}{dt}\right)^2\right]^2} = \frac{-\rho_0 b_2}{3\sqrt{a_2(t_2-t)^{\frac{2}{3}}+b_2}\left[(1+a_2r^2)(t_2-t)^{\frac{2}{3}}+b_2r^2\right]^2} \tag{4.34}$$

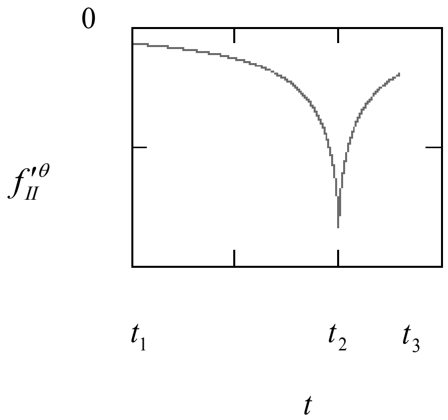

Fig. 6. Plotting $f_{II}'^{\theta}$ versus t with fixed r during the second leg of time travel.

It is convenient to write Equations (4.33) and (4.34) as a function of time t' measured in S′. For this, it is necessary to find t as a function of t' from Equation (4.23)

$$(1+a_2r^2)(t_2-t)^{\frac{2}{3}}+b_2r^2 = \left[(1+a_2r^2)(t_1'-t')+\left[(1+a_2r^2)(t_2-t_1)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}} \tag{4.35}$$

$$(t_2-t)^{\frac{2}{3}} = \frac{1}{1+a_2r^2}\left[(1+a_2r^2)(t_1'-t')+\left[(1+a_2r^2)(t_2-t_1)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}} - \frac{b_2r^2}{1+a_2r^2} \tag{4.36}$$

$$t = t_2 \pm \left\{\frac{1}{1+a_2r^2}\left[(1+a_2r^2)(t_1'-t')+\left[(1+a_2r^2)(t_2-t_1)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}} - \frac{b_2r^2}{1+a_2r^2}\right\}^{\frac{3}{2}} \tag{4.37}$$

with t_1' given by Equation (4.26).
Introducing Equations (4.35) and (4.36) into Equations (4.33) and (4.34) we obtain

$$f_{II}'^{r} = -\rho_0 r \frac{a_2\left[\left(1+a_2r^2\right)\left(t_1'-t'\right)+\left[\left(1+a_2r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}}+b_2}{\left(1+a_2r^2\right)\left[\left(1+a_2r^2\right)\left(t_1'-t'\right)+\left[\left(1+a_2r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}}} \tag{4.38}$$

$$f_{II}'^{\theta} = \frac{-\rho_0 b_2\sqrt{1+a_2r^2}}{3\sqrt{a_2\left[\left(1+a_2r^2\right)\left(t_1'-t'\right)+\left[\left(1+a_2r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}}+b_2}\left[\left(1+a_2r^2\right)\left(t_1'-t'\right)+\left[\left(1+a_2r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}\right]^{\frac{4}{3}}} \tag{4.39}$$

where t_1' is given by Equation (4.26). We also note that $f_{II}'^{r}$ and $f_{II}'^{\theta}$ are finite through the interval.

The fluid reaches the speed of light in S′ during this leg of time travel. This can be checked by inserting Equation (4.16) into Equation (3.24)

$$v_{II}' = \frac{r\frac{d\varphi_{II}}{dt}}{\sqrt{1+r^2\left(\frac{d\varphi_{II}}{dt}\right)^2}} = \frac{r\sqrt{a_2\left(t_2-t\right)^{\frac{2}{3}}+b_2}}{\sqrt{\left(1+a_2r^2\right)\left(t_2-t\right)^{\frac{2}{3}}+b_2r^2}} \tag{4.40}$$

so that

$$v_{II}'(t_2) = 1 \tag{4.41}$$

at time $t = t_2$ in S or at time t_2' in S′ where

$$t_2' = t_{II}'(t_2) = t_1' - \frac{1}{1+a_2r^2}b_2^{\frac{3}{2}}r^3 + \frac{1}{1+a_2r^2}\left[\left(1+a_2r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}} \tag{4.42}$$

is obtained by doing $t = t_2$ in Equation (4.23) and t_1' is given by Equation (4.26).

It is striking that a fluid can reach the speed of light in S′. This is not possible in the case of a material particle. The difference is in the equation of continuity (1.19) which is only verified in the case of the fluid. This equation shows that the proper energy density of the fluid obtained from Equation (3.13) with Equations (4.30), (4.3), (4.20) and (4.16) during the second leg of the travel

$$\rho_{II} = \frac{\rho_0}{1+r^2\left(\frac{d\varphi_{II}}{dt}\right)^2} = \frac{\rho_0\left(t_2-t\right)^{\frac{2}{3}}}{\left(1+a_2r^2\right)\left(t_2-t\right)^{\frac{2}{3}}+b_2r^2} \tag{4.43}$$

can be null at time $t = t_2$ in S in which the fluid reaches the velocity of light in S′ and time reversal occurs. This happens gradually in S′ at time t_2' given by Equation (4.42) because t_2' depends on r. As mass-energy is conserved in S′ by checking the continuity equation (1.19), this means that the mass of the fluid is progressively transformed into energy. This can happen instantaneously in S′ if the fluid is confined to a ring of infinitesimal width. It is due to this effect that the fluid can reach the speed of light in the reference frame S′.

Back to the past

The third leg of time travel occurs during the interval $t_3 \le t \le t_4$ for an observer located in S. It is convenient to keep angular velocity $\frac{d\varphi}{dt}$ constant during this leg. As the angular velocity must be continuous this means that

$$\left.\frac{d\varphi_{III}}{dt}\right|_{t=t_3} = \left.\frac{d\varphi_{II}}{dt}\right|_{t=t_3} \tag{4.44}$$

and also that according to Equations (4.44) and (4.16) that

$$\frac{d\varphi_{III}}{dt} = \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{a_2(t_2 - t_3)^{\frac{2}{3}} + b_2} \tag{4.45}$$

Integrating Equation (4.45) we obtain

$$\varphi_{III}(t) = \varphi_{III}(t_3) + \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{a_2(t_2 - t_3)^{\frac{2}{3}} + b_2}\,(t - t_3) \tag{4.46}$$

Continuity of function $\varphi(t)$ implies

$$\varphi_{III}(t_3) = \varphi_{II}(t_3) = \frac{a_1}{2}(t_1 - t_0)^2 - \frac{1}{a_2}\left[a_2(t_2 - t_3)^{\frac{2}{3}} + b_2\right]^{\frac{3}{2}} + \frac{1}{a_2}\left[a_2(t_2 - t_1)^{\frac{2}{3}} + b_2\right]^{\frac{3}{2}} \tag{4.47}$$

as can be deduced from Equations (4.14) and (4.19).
Deriving Equation (4.45) we obtain the angular acceleration

$$\frac{d^2\varphi_{III}}{dt^2} = 0 \tag{4.48}$$

Introducing Equation (4.45) into Equation (2.5)

$$\frac{\partial t'_{III}}{\partial t} = -\sqrt{1 + r^2\left(\frac{d\varphi_{II}}{dt}\right)^2} = \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2} \tag{4.49}$$

and integrating

$$t'_{III} = t'_3 + \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2}\,(t - t_3) \tag{4.50}$$

Continuity of t' implies

$$t'_{III}(t_3) = t'_{II}(t_3) \tag{4.51}$$

that with Equation (4.50) gives

$$t'_3 = t'_{II}(t_3) \tag{4.52}$$

Introducing Equation (4.23) at $t = t_3$ into Equation (4.52) and making use of Equation (4.26)

$$t'_3 = t'_0 + \frac{t_1 - t_0}{2}\sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2}\right]$$
$$- \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} + \frac{1}{1 + a_2 r^2}\left[(1 + a_2 r^2)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} \tag{4.53}$$

Substituting Equation (4.53) into Equation (4.50) yields

$$t'_{III} = t'_0 + \frac{t_1 - t_0}{2}\sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1^2 r^2 (t_1 - t_0)^2}\right]$$

$$-\frac{1}{1+a_2r^2}\left[\left(1+a_2r^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}+\frac{1}{1+a_2r^2}\left[\left(1+a_2r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}$$

$$+\frac{1}{\left(t_2-t_3\right)^{\frac{1}{3}}}\sqrt{\left(1+a_2r^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r^2}\left(t-t_3\right) \tag{4.54}$$

Deriving Equation (4.50) and making use of Equations (4.52) and (4.29) at $t=t_3$ we obtain

$$\frac{\partial t'_{III}}{\partial r}=\frac{t_1-t_0}{2r}\sqrt{1+a_1{}^2r^2\left(t_1-t_0\right)^2}-\frac{1}{2a_1r^2}Ln\left[a_1r\left(t_1-t_0\right)+\sqrt{1+a_1{}^2r^2\left(t_1-t_0\right)^2}\right]$$

$$-\frac{a_2r\left(1+a_2r^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+a_2b_2r^3+3b_2r}{\left(1+a_2r^2\right)^2}\sqrt{\left(1+a_2r^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r^2}$$

$$+\frac{a_2r\left(1+a_2r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+a_2b_2r^3+3b_2r}{\left(1+a_2r^2\right)^2}\sqrt{\left(1+a_2r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r^2}$$

$$+\frac{a_2r\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r}{\left(t_2-t_3\right)^{\frac{1}{3}}\sqrt{\left(1+a_2r^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r^2}}\left(t-t_3\right) \tag{4.55}$$

It is necessary to determine the velocity (4.49) of the travel backward in time. As $\frac{\partial t'_{III}}{\partial t}$ depends on r to determine this velocity it is convenient to fix r so that, in $r=r_0$ verifies

$$\left.\frac{\partial t'_{III}}{\partial t}\right|_{r_0,t_3}=-k \tag{4.56}$$

where k is a constant greater than zero

$$k>0 \tag{4.57}$$

Inserting Equation (4.49) into Equation (4.56) yields

$$\frac{1}{\left(t_2-t_3\right)^{\frac{1}{3}}}\sqrt{\left(1+a_2r_0{}^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r_0{}^2}=-k \tag{4.58}$$

$$\left(1+a_2r_0{}^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r_0{}^2=k^2\left(t_2-t_3\right)^{\frac{2}{3}} \tag{4.59}$$

$$t_3=t_2+\left(\frac{b_2r_0{}^2}{k^2-1-a_2r_0{}^2}\right)^{\frac{3}{2}} \tag{4.60}$$

which are the same equation.

The proper energy density at time $t=t_3$ is obtained from Equation (3.13), making use of Equations (4.30) and (4.20)

$$\rho_3=\rho(t_3)=\frac{\rho_0}{1+r^2\left(\frac{d\varphi_{II}}{dt}\right)^2_{t=t_3}} \tag{4.61}$$

Introducing Equations (4.45), (4.48) and (4.61) into Equations (3.16) and (3.17) and making use of Equation (4.44) we obtain for this fluid in the reference frame S during the third leg of time travel

$$f_{III}^{r} = -\rho_0 r \frac{\left(\frac{d\varphi_{III}}{dt}\right)^2}{1 + r^2\left(\frac{d\varphi_{III}}{dt}\right)^2} = -\rho_0 r \frac{a_2(t_2 - t_3)^{\frac{2}{3}} + b_2}{(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2} \tag{4.62}$$

$$f_{III}^{\theta} = \rho_0 \frac{r\sqrt{1 + r^2\left(\frac{d\varphi_{III}}{dt}\right)^2}\,\frac{\partial t'_{III}}{\partial r}\left(\frac{d\varphi_{III}}{dt}\right)^3 - \frac{d^2\varphi_{III}}{dt^2}}{\left[1 + r^2\left(\frac{d\varphi_{III}}{dt}\right)^2\right]^2}$$

$$= \rho_0 \frac{r\sqrt{(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2}\,\frac{\partial t'_{III}}{\partial r}\left[a_2(t_2 - t_3)^{\frac{2}{3}} + b_2\right]^{\frac{3}{2}}}{\left[(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2\right]^2} \tag{4.63}$$

where $\frac{\partial t'_{III}}{\partial r}$ is given by Equation (4.55).

The third leg of time travel occurs during the interval $t'_3 \le t' \le t'_4$ for an observer located in S′. The rotating motion of this fluid in this reference frame during this leg is obtained with Equations (3.21) and (4.62) and introducing Equation (4.48) into Equation (3.22)

$$f_{III}^{\prime r} = f_{III}^{r} = -\rho_0 r \frac{a_2(t_2 - t_3)^{\frac{2}{3}} + b_2}{(1 + a_2 r^2)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2} \tag{4.64}$$

$$f_{III}^{\prime\theta} = -\rho_3 \frac{\left[1 + r^2\left(\frac{d\varphi_{III}}{dt}\right)^2_{t=t_3}\right]\frac{d^2\varphi_{III}}{dt^2}}{\left[1 + r^2\left(\frac{d\varphi_{III}}{dt}\right)^2\right]^2} = 0 \tag{4.65}$$

New time reversal

The fourth leg of time travel occurs during the interval $t_4 \le t \le t_6$ for an observer located in S. The simplest function $\varphi(t)$ which gives a new time reversal is

$$\varphi_{IV}(t) = \varphi_{IV}(t_4) + \frac{1}{a_4}\left[a_4(t - t_5)^{\frac{2}{3}} + b_4\right]^{\frac{3}{2}} - \frac{1}{a_4}\left[a_4(t_4 - t_5)^{\frac{2}{3}} + b_4\right]^{\frac{3}{2}} \tag{4.66}$$

where t_5, a_4 and b_4 are constants verifying

$$t_4 < t_5 < t_6;\;\; b_4 > 0;\;\; b_4 > -a_4(t_4 - t_5)^{\frac{2}{3}} \tag{4.67}$$

The angular velocity is

$$\frac{d\varphi_{IV}}{dt} = \frac{1}{(t - t_5)^{\frac{1}{3}}}\sqrt{a_4(t - t_5)^{\frac{2}{3}} + b_4} \tag{4.68}$$

The angular acceleration is

$$\frac{d^2\varphi_{IV}}{dt^2} = -\frac{b_4}{3(t-t_5)^{\frac{4}{3}}\sqrt{a_4(t-t_5)^{\frac{2}{3}}+b_4}} \tag{4.69}$$

The functions $\varphi(t)$ and $\frac{d\varphi}{dt}$ must be continuous at $t = t_4$

$$\varphi_{IV}(t_4) = \varphi_{III}(t_4) \tag{4.70}$$

that with Equations (4.46), (4.47) and (4.66) gives

$$\varphi_{IV}(t_4) = \frac{a_1}{2}(t_1-t_0)^2 - \frac{1}{a_2}\left[a_2(t_2-t_3)^{\frac{2}{3}}+b_2\right]^{\frac{3}{2}} + \frac{1}{a_2}\left[a_2(t_2-t_1)^{\frac{2}{3}}+b_2\right]^{\frac{3}{2}}$$

$$+\frac{1}{(t_2-t_3)^{\frac{1}{3}}}\sqrt{a_2(t_2-t_3)^{\frac{2}{3}}+b_2}\,(t_4-t_3) \tag{4.71}$$

and also

$$\left.\frac{d\varphi_{III}}{dt}\right|_{t=t_4} = \left.\frac{d\varphi_{IV}}{dt}\right|_{t=t_4} \tag{4.72}$$

The last equation with Equations (4.45) and (4.68) leads to

$$\frac{1}{(t_2-t_3)^{\frac{1}{3}}}\sqrt{a_2(t_2-t_3)^{\frac{2}{3}}+b_2} = \frac{1}{(t_4-t_5)^{\frac{1}{3}}}\sqrt{a_4(t_4-t_5)^{\frac{2}{3}}+b_4} \tag{4.73}$$

Introducing Equation (4.68) into Equation (2.5) gives

$$\frac{\partial t'_{IV}}{\partial t} = \mp\sqrt{1+r^2\left(\frac{d\varphi_{IV}}{dt}\right)^2} = \frac{1}{(t-t_5)^{\frac{1}{3}}}\sqrt{(1+a_4r^2)(t-t_5)^{\frac{2}{3}}+b_4r^2} \tag{4.74}$$

It is observed that $\frac{\partial t'_{IV}}{\partial t}$ changes sign at $t = t_5$ as it was discussed in Equation (2.5).

Integrating Equation (4.74) we obtain

$$t'_{IV} = t'_4 + \int_{t_4}^{t}\frac{1}{(t-t_5)^{\frac{1}{3}}}\sqrt{(1+a_4r^2)(t-t_5)^{\frac{2}{3}}+b_4r^2}\,dt$$

$$= t'_4 + \frac{1}{1+a_4r^2}\left[(1+a_4r^2)(t-t_5)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}} - \frac{1}{1+a_4r^2}\left[(1+a_4r^2)(t_4-t_5)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}} \tag{4.75}$$

Continuity of t' implies

$$t'_{IV}(t_4) = t'_{III}(t_4) \tag{4.76}$$

which with Equation (4.75) gives

$$t'_4 = t'_{III}(t_4) \tag{4.77}$$

Introducing Equation (4.50) at $t = t_4$ into Equation (4.77) and making use of Equation (4.53) we obtain

$$t'_4 = t'_0 + \frac{t_1-t_0}{2}\sqrt{1+a_1^2r^2(t_1-t_0)^2} + \frac{1}{2a_1r}Ln\left[a_1r(t_1-t_0)+\sqrt{1+a_1^2r^2(t_1-t_0)^2}\right]$$

$$-\frac{1}{1+a_2r^2}\left[(1+a_2r^2)(t_2-t_3)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}} + \frac{1}{1+a_2r^2}\left[(1+a_2r^2)(t_2-t_1)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}$$

$$+\frac{1}{(t_2-t_3)^{\frac{1}{3}}}\sqrt{\left(1+a_2r^2\right)(t_2-t_3)^{\frac{2}{3}}+b_2r^2}\,(t_4-t_3) \tag{4.78}$$

Introducing Equation (4.78) into Equation (4.75) yields

$$t'_{IV}=t'_0+\frac{t_1-t_0}{2}\sqrt{1+a_1^2r^2(t_1-t_0)^2}+\frac{1}{2a_1r}Ln\left[a_1r(t_1-t_0)+\sqrt{1+a_1^2r^2(t_1-t_0)^2}\right]$$

$$-\frac{1}{1+a_2r^2}\left[\left(1+a_2r^2\right)(t_2-t_3)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}+\frac{1}{1+a_2r^2}\left[\left(1+a_2r^2\right)(t_2-t_1)^{\frac{2}{3}}+b_2r^2\right]^{\frac{3}{2}}$$

$$+\frac{1}{(t_2-t_3)^{\frac{1}{3}}}\sqrt{\left(1+a_2r^2\right)(t_2-t_3)^{\frac{2}{3}}+b_2r^2}\,(t_4-t_3)+\frac{1}{1+a_4r^2}\left[\left(1+a_4r^2\right)(t-t_5)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}}$$

$$-\frac{1}{1+a_4r^2}\left[\left(1+a_4r^2\right)(t_4-t_5)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}} \tag{4.79}$$

In order to avoid divergences in Equation (4.79) it is convenient that

$$a_4>0 \tag{4.80}$$

The form of the function (4.79) that reaches a minimum at $t=t_5$ allows a new time reversal. At time $t=t_5$ in S, the new time reversal occurs. Again, this is only possible if the density of the external force given by Equations (3.16) and (3.17) applied on this fluid in S and (3.21) and (3.22) in S′ are finite.

Deriving Equation (4.75), making use of Equations (4.77) and (4.55) at $t=t_4$ we obtain

$$\frac{\partial t'_{IV}}{\partial r}=\frac{t_1-t_0}{2r}\sqrt{1+a_1^2r^2(t_1-t_0)^2}-\frac{1}{2a_1r^2}Ln\left[a_1r(t_1-t_0)+\sqrt{1+a_1^2r^2(t_1-t_0)^2}\right]$$

$$-\frac{a_2r\left(1+a_2r^2\right)(t_2-t_3)^{\frac{2}{3}}+a_2b_2r^3+3b_2r}{\left(1+a_2r^2\right)^2}\sqrt{\left(1+a_2r^2\right)(t_2-t_3)^{\frac{2}{3}}+b_2r^2}$$

$$+\frac{a_2r\left(1+a_2r^2\right)(t_2-t_1)^{\frac{2}{3}}+a_2b_2r^3+3b_2r}{\left(1+a_2r^2\right)^2}\sqrt{\left(1+a_2r^2\right)(t_2-t_1)^{\frac{2}{3}}+b_2r^2}$$

$$+\frac{a_2r(t_2-t_3)^{\frac{2}{3}}+b_2r}{(t_2-t_3)^{\frac{1}{3}}\sqrt{\left(1+a_2r^2\right)(t_2-t_3)^{\frac{2}{3}}+b_2r^2}}(t_4-t_3)$$

$$-\frac{a_4r\left(1+a_4r^2\right)(t_4-t_5)^{\frac{2}{3}}+a_4b_4r^3+3b_4r}{\left(1+a_4r^2\right)^2}\sqrt{\left(1+a_4r^2\right)(t_4-t_5)^{\frac{2}{3}}+b_4r^2}$$

$$+\frac{a_4r\left(1+a_4r^2\right)(t-t_5)^{\frac{2}{3}}+a_4b_4r^3+3b_4r}{\left(1+a_4r^2\right)^2}\sqrt{\left(1+a_4r^2\right)(t-t_5)^{\frac{2}{3}}+b_4r^2}<0 \tag{4.81}$$

The proper energy density at time $t=t_4$ is calculated from Equations (3.13), (4.61) and (4.44)

$$\rho_4 = \rho(t_4) = \frac{\rho_0}{1 + r^2\left(\frac{d\varphi_{III}}{dt}\right)^2_{t=t_4}} \tag{4.82}$$

Introducing Equations (4.68), (4.69) and (4.82) into Equations (3.16) and (3.17) and making use of Equation (4.72) we obtain for this fluid in the reference frame S during the fourth leg of time travel

$$f_{IV}^r = -\rho_0 r \frac{\left(\frac{d\varphi_{IV}}{dt}\right)^2}{1 + r^2\left(\frac{d\varphi_{IV}}{dt}\right)^2} = -\rho_0 r \frac{a_4(t - t_5)^{\frac{2}{3}} + b_4}{\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2} \tag{4.83}$$

$$f_{IV}^\theta = \rho_0 \frac{\mp r\sqrt{1 + r^2\left(\frac{d\varphi_{IV}}{dt}\right)^2}\,\frac{\partial t'_{IV}}{\partial r}\left(\frac{d\varphi_{IV}}{dt}\right)^3 - \frac{d^2\varphi_{IV}}{dt^2}}{\left[1 + r^2\left(\frac{d\varphi_{IV}}{dt}\right)^2\right]^2}$$

$$= \rho_0 \frac{r\sqrt{\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2}\,\frac{\partial t'_{IV}}{\partial r}\left[a_4(t - t_5)^{\frac{2}{3}} + b_4\right]^{\frac{3}{2}} + \frac{b_4}{3\sqrt{a_4(t - t_5)^{\frac{2}{3}} + b_4}}}{\left[\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2\right]^2} \tag{4.84}$$

where $\frac{\partial t'_{IV}}{\partial r}$ is given by Equation (4.81). We also note that f_{IV}^r and f_{IV}^θ are finite through the interval.

The fourth leg of time travel occurs during the interval $t'_4 \le t' \le t'_6$ for an observer located in S´. The rotating motion of this fluid in this reference frame during this leg is obtained with Equations (3.21) and (4.83), introducing Equations (4.68), (4.69) and (4.82) into Equation (3.22) and taking into account Equation (4.72)

$$f_{IV}'^r = f_{IV}^r = -\rho_0 r \frac{a_4(t - t_5)^{\frac{2}{3}} + b_4}{\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2} \tag{4.85}$$

$$f_{IV}'^\theta = -\rho_0 \frac{\frac{d^2\varphi_{IV}}{dt^2}}{\left[1 + r^2\left(\frac{d\varphi_{IV}}{dt}\right)^2\right]^2} = \frac{\rho_0 b_4}{3\sqrt{a_4(t - t_5)^{\frac{2}{3}} + b_4}\left[\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2\right]^2} \tag{4.86}$$

It is convenient to write Equations (4.85) and (4.86) as a function of time t' measured in S´. For this, it is necessary to find t as a function of t' from Equation (4.75)

$$\left(1 + a_4 r^2\right)(t - t_5)^{\frac{2}{3}} + b_4 r^2 = \left[\left(1 + a_4 r^2\right)(t' - t'_4) + \left[\left(1 + a_4 r^2\right)(t_4 - t_5)^{\frac{2}{3}} + b_4 r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}} \tag{4.87}$$

$$\left(t-t_5\right)^{\frac{2}{3}}=\frac{1}{1+a_4r^2}\left[\left(1+a_4r^2\right)\left(t'-t'_4\right)+\left[\left(1+a_4r^2\right)\left(t_4-t_5\right)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}}-\frac{b_4r^2}{1+a_4r^2} \tag{4.88}$$

$$t=t_5\pm\left\{\frac{1}{1+a_4r^2}\left[\left(1+a_4r^2\right)\left(t'-t'_4\right)+\left[\left(1+a_4r^2\right)\left(t_4-t_5\right)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}}-\frac{b_4r^2}{1+a_4r^2}\right\}^{\frac{3}{2}} \tag{4.89}$$

with t'_4 given by Equation (4.78).

Introducing Equations (4.87) and (4.88) into Equations (4.85) and (4.86) we obtain

$$f'^{r}_{IV}=-\rho_0 r\frac{a_4\left[\left(1+a_4r^2\right)\left(t'-t'_4\right)+\left[\left(1+a_4r^2\right)\left(t_4-t_5\right)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}}+b_4}{\left(1+a_4r^2\right)\left[\left(1+a_4r^2\right)\left(t'-t'_4\right)+\left[\left(1+a_4r^2\right)\left(t_4-t_5\right)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}}} \tag{4.90}$$

$$f'^{\theta}_{IV}=\frac{\rho_0 b_4\sqrt{1+a_4r^2}}{3\sqrt{a_4\left[\left(1+a_4r^2\right)\left(t'-t'_4\right)+\left[\left(1+a_4r^2\right)\left(t_4-t_5\right)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}}\right]^{\frac{2}{3}}+b_4}\left[\left(1+a_4r^2\right)\left(t'-t'_4\right)+\left[\left(1+a_4r^2\right)\left(t_4-t_5\right)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}}\right]^{\frac{4}{3}}} \tag{4.91}$$

where t'_4 is given by Equation (4.78). We also note that f'^{r}_{IV} and f'^{θ}_{IV} are finite through the interval.
Again, and like at time $t=t_2$, it can be verified that at time $t=t_5$ in S, the fluid reaches the speed of light in S´. This happens in S´ at time t'_5 obtained by doing $t=t_5$ in Equation (4.75)

$$t'_5=t'_{IV}(t_5)=t'_4+\frac{1}{1+a_4r^2}b_4^{\frac{3}{2}}r^3-\frac{1}{1+a_4r^2}\left[\left(1+a_4r^2\right)\left(t_4-t_5\right)^{\frac{2}{3}}+b_4r^2\right]^{\frac{3}{2}} \tag{4.92}$$

where t'_4 is given by Equation (4.78).

Time deceleration

The fifth leg of time travel occurs during the interval $t_6\le t\le T$ for an observer located in S. If the rotation stops at the end of the travel, at time $t=T$ in S, the simplest function $\varphi(t)$ is

$$\varphi_V(t)=\varphi_V(t_6)+\frac{a_5}{2}\left(t-T\right)^2-\frac{a_5}{2}\left(t_6-T\right)^2 \tag{4.93}$$

The angular velocity is

$$\frac{d\varphi_V}{dt}=a_5\left(t-T\right) \tag{4.94}$$

which is zero when the motion stops at the end

$$\left.\frac{d\varphi_V}{dt}\right|_{t=T} = 0 \tag{4.95}$$

The angular acceleration is

$$\frac{d^2\varphi_V}{dt^2} = a_5 \tag{4.96}$$

where a_5 is a constant such that

$$a_5 < 0 \tag{4.97}$$

The functions $\varphi(t)$ and $\frac{d\varphi}{dt}$ must be continuous at $t = t_6$

$$\varphi_V(t_6) = \varphi_{IV}(t_6) \tag{4.98}$$

which with Equations (4.93), (4.66) and (4.71) leads to

$$\varphi_V(t_6) = \frac{a_1}{2}(t_1 - t_0)^2 - \frac{1}{a_2}\left[a_2(t_2 - t_3)^{\frac{2}{3}} + b_2\right]^{\frac{3}{2}} + \frac{1}{a_2}\left[a_2(t_2 - t_1)^{\frac{2}{3}} + b_2\right]^{\frac{3}{2}}$$
$$+ \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{a_2(t_2 - t_3)^{\frac{2}{3}} + b_2}\,(t_4 - t_3) + \frac{1}{a_4}\left[a_4(t_6 - t_5)^{\frac{2}{3}} + b_4\right]^{\frac{3}{2}}$$
$$- \frac{1}{a_4}\left[a_4(t_4 - t_5)^{\frac{2}{3}} + b_4\right]^{\frac{3}{2}} \tag{4.99}$$

and

$$\left.\frac{d\varphi_V}{dt}\right|_{t=t_6} = \left.\frac{d\varphi_{IV}}{dt}\right|_{t=t_6} \tag{4.100}$$

The last equation together with Equations (4.94) and (4.68) gives

$$a_5(t_6 - T) = \frac{1}{(t_6 - t_5)^{\frac{1}{3}}}\sqrt{a_4(t_6 - t_5)^{\frac{2}{3}} + b_4} \tag{4.101}$$

Inserting Equation (4.94) into Equation (2.5)

$$\frac{\partial t'_V}{\partial t} = \sqrt{1 + r^2\left(\frac{d\varphi_V}{dt}\right)^2} = \sqrt{1 + a_5{}^2 r^2 (t - T)^2} \tag{4.102}$$

and integrating we obtain

$$t'_V = t'_6 + \int_{t_6}^{t}\sqrt{1 + a_5{}^2 r^2 (t - T)^2}\,dt = t'_6 + \frac{t - T}{2}\sqrt{1 + a_5{}^2 r^2 (t - T)^2} + \frac{1}{2a_5 r} Ln\left[a_5 r(t - T) + \sqrt{1 + a_5{}^2 r^2 (t - T)^2}\right]$$
$$- \frac{t_6 - T}{2}\sqrt{1 + a_5{}^2 r^2 (t_6 - T)^2} - \frac{1}{2a_5 r} Ln\left[a_5 r(t_6 - T) + \sqrt{1 + a_5{}^2 r^2 (t_6 - T)^2}\right] \tag{4.103}$$

Continuity of t' implies

$$t'_V(t_6) = t'_{IV}(t_6) \tag{4.104}$$

which with Equation (4.103) gives

$$t'_6 = t'_{IV}(t_6) \tag{4.105}$$

Introducing Equation (4.75) at $t = t_6$ into Equation (4.105) and making use of Equation (4.78) we obtain

$$t_6' = t_0' + \frac{t_1 - t_0}{2}\sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2}\right]$$

$$- \frac{1}{1 + a_2 r^2}\left[\left(1 + a_2 r^2\right)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} + \frac{1}{1 + a_2 r^2}\left[\left(1 + a_2 r^2\right)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}}$$

$$+ \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{\left(1 + a_2 r^2\right)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2}\,(t_4 - t_3) + \frac{1}{1 + a_4 r^2}\left[\left(1 + a_4 r^2\right)(t_6 - t_5)^{\frac{2}{3}} + b_4 r^2\right]^{\frac{3}{2}}$$

$$- \frac{1}{1 + a_4 r^2}\left[\left(1 + a_4 r^2\right)(t_4 - t_5)^{\frac{2}{3}} + b_4 r^2\right]^{\frac{3}{2}} \quad (4.106)$$

The above expression is simplified if the constants are chosen so that

$$\begin{aligned} t_5 - t_4 &= t_3 - t_2 \\ t_6 - t_5 &= t_2 - t_1 \\ a_4 &= a_2 \\ b_4 &= b_2 \end{aligned} \quad (4.107)$$

Introducing Equation (4.107) into Equation (4.106) gives

$$t_6' = t_0' + \frac{t_1 - t_0}{2}\sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2}\right]$$

$$- \frac{2}{1 + a_2 r^2}\left[\left(1 + a_2 r^2\right)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} + \frac{2}{1 + a_2 r^2}\left[\left(1 + a_2 r^2\right)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}}$$

$$+ \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{\left(1 + a_2 r^2\right)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2}\,(t_4 - t_3) \quad (4.108)$$

Substituting Equation (4.108) into Equation (4.103) yields

$$t_V' = t_0' + \frac{t_1 - t_0}{2}\sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r} Ln\left[a_1 r(t_1 - t_0) + \sqrt{1 + a_1{}^2 r^2 (t_1 - t_0)^2}\right]$$

$$- \frac{2}{1 + a_2 r^2}\left[\left(1 + a_2 r^2\right)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}} + \frac{2}{1 + a_2 r^2}\left[\left(1 + a_2 r^2\right)(t_2 - t_1)^{\frac{2}{3}} + b_2 r^2\right]^{\frac{3}{2}}$$

$$+ \frac{1}{(t_2 - t_3)^{\frac{1}{3}}}\sqrt{\left(1 + a_2 r^2\right)(t_2 - t_3)^{\frac{2}{3}} + b_2 r^2}\,(t_4 - t_3) + \frac{t - T}{2}\sqrt{1 + a_5{}^2 r^2 (t - T)^2}$$

$$+ \frac{1}{2a_5 r} Ln\left[a_5 r(t - T) + \sqrt{1 + a_5{}^2 r^2 (t - T)^2}\right] - \frac{t_6 - T}{2}\sqrt{1 + a_5{}^2 r^2 (t_6 - T)^2}$$

$$- \frac{1}{2a_5 r} Ln\left[a_5 r(t_6 - T) + \sqrt{1 + a_5{}^2 r^2 (t_6 - T)^2}\right] \quad (4.109)$$

It is convenient that the travel ends at a certain time T' in the past of S´. As $t' = t'(t, r)$, it is advisable to fix r in which this happens. This will be $r = r_0$ and so

$$t_V'(T)_{r_0} = T' \quad (4.110)$$

which with Equation (4.109) gives

$$T' = t_0' + \frac{t_1 - t_0}{2}\sqrt{1 + a_1{}^2 r_0{}^2 (t_1 - t_0)^2} + \frac{1}{2a_1 r_0} Ln\left[a_1 r_0 (t_1 - t_0) + \sqrt{1 + a_1{}^2 r_0{}^2 (t_1 - t_0)^2}\right]$$

$$-\frac{2}{1+a_2r_0^2}\left[\left(1+a_2r_0^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r_0^2\right]^{\frac{3}{2}}+\frac{2}{1+a_2r_0^2}\left[\left(1+a_2r_0^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r_0^2\right]^{\frac{3}{2}}$$

$$+\frac{1}{\left(t_2-t_3\right)^{\frac{1}{3}}}\sqrt{\left(1+a_2r_0^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r_0^2}\left(t_4-t_3\right)-\frac{t_6-T}{2}\sqrt{1+a_5^2r_0^2\left(t_6-T\right)^2}$$

$$-\frac{1}{2a_5r_0}Ln\left[a_5r_0\left(t_6-T\right)+\sqrt{1+a_5^2r_0^2\left(t_6-T\right)^2}\right] \tag{4.111}$$

This equation is simplified if the constants are chosen so that

$$T-t_6=t_1-t_0$$
$$a_5=-a_1 \tag{4.112}$$

Introducing Equation (4.112) into Equation (4.111) we obtain

$$T'=t_0'+\left(t_1-t_0\right)\sqrt{1+a_1^2r_0^2\left(t_1-t_0\right)^2}+\frac{1}{a_1r_0}Ln\left[a_1r_0\left(t_1-t_0\right)+\sqrt{1+a_1^2r_0^2\left(t_1-t_0\right)^2}\right]$$

$$-\frac{2}{1+a_2r_0^2}\left[\left(1+a_2r_0^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r_0^2\right]^{\frac{3}{2}}+\frac{2}{1+a_2r_0^2}\left[\left(1+a_2r_0^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r_0^2\right]^{\frac{3}{2}}$$

$$+\frac{1}{\left(t_2-t_3\right)^{\frac{1}{3}}}\sqrt{\left(1+a_2r_0^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2r_0^2}\left(t_4-t_3\right) \tag{4.113}$$

which with Equations (4.58) and (4.59) can be written as

$$T'=t_0'+\left(t_1-t_0\right)\sqrt{1+a_1^2r_0^2\left(t_1-t_0\right)^2}+\frac{1}{a_1r_0}Ln\left[a_1r_0\left(t_1-t_0\right)+\sqrt{1+a_1^2r_0^2\left(t_1-t_0\right)^2}\right]$$

$$-\frac{2}{1+a_2r_0^2}k^3\left(t_3-t_2\right)+\frac{2}{1+a_2r_0^2}\left[\left(1+a_2r_0^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r_0^2\right]^{\frac{3}{2}}-k\left(t_4-t_3\right) \tag{4.114}$$

It is convenient to introduce a new parameter δ defined as

$$a_1r_0\left(t_1-t_0\right)=\delta \tag{4.115}$$

In summary, the boundary conditions (4.21) and (4.60) and Equations (4.107), (4.112) , (4.114) and (4.115) provide ten independent equations

$$a_1=\frac{1}{t_1-t_0}\sqrt{a_2+\frac{b_2}{\left(t_2-t_1\right)^{\frac{2}{3}}}}\;;\; t_3=t_2+\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}}\;;\; t_5-t_4=t_3-t_2\;;$$

$$t_6-t_5=t_2-t_1;\; a_4=a_2;\; b_4=b_2;\; T-t_6=t_1-t_0;\; a_5=-a_1;\; a_1r_0\left(t_1-t_0\right)=\delta$$

$$T'=t_0'+\left(t_1-t_0\right)\sqrt{1+a_1^2r_0^2\left(t_1-t_0\right)^2}+\frac{1}{a_1r_0}Ln\left[a_1r_0\left(t_1-t_0\right)+\sqrt{1+a_1^2r_0^2\left(t_1-t_0\right)^2}\right]$$

$$-\frac{2}{1+a_2r_0^2}k^3\left(t_3-t_2\right)+\frac{2}{1+a_2r_0^2}\left[\left(1+a_2r_0^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2r_0^2\right]^{\frac{3}{2}}-k\left(t_4-t_3\right) \tag{4.116}$$

because Equations (4.73) and (4.101) are identically satisfied. In addition, there are 13 unknowns which leaves three free parameters, for example, a_2, b_2 and δ. The

unknowns will be functions of k, r_0, t_0, t_0', T and T', that are known, plus the free parameters a_2, b_2 and δ.
Solving these equations we obtain

$$t_1 = t_0 + \frac{kT - t_0' + T' - kt_0 - 2k\left(1-\frac{k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}} - 2\left(\frac{1+\delta^2}{1+a_2r_0^2}+k\right)\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}}{2k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)} \quad (4.117)$$

$$t_2 = t_0 + \frac{kT - t_0' + T' - kt_0 - 2k\left(1-\frac{k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}} - 2\left(\frac{1+\delta^2}{1+a_2r_0^2}+k\right)\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}}{2k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)} + \left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}} \quad (4.118)$$

$$t_3 = t_0 + \frac{kT - t_0' + T' - kt_0 - 2k\left(1-\frac{k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}} - 2\left(\frac{1+\delta^2}{1+a_2r_0^2}+k\right)\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}}{2k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)} + \left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}} + \left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}} \quad (4.119)$$

$$t_4 = t_0 + \frac{t_0'-T'}{k} + \left(1-\frac{2k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}} + \frac{1}{k}\left[\frac{2\left(1+\delta^2\right)}{1+a_2r_0^2}+k\right]\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}$$
$$+\frac{T-\frac{t_0'-T'}{k}-t_0-2\left(1-\frac{k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}}-\frac{2}{k}\left(\frac{1+\delta^2}{1+a_2r_0^2}+k\right)\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}}{2k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)}\left[k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)\right] \quad (4.120)$$

$$t_5 = t_0 + \frac{t_0'-T'}{k} + 2\left(1-\frac{k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}} + \frac{1}{k}\left[\frac{2\left(1+\delta^2\right)}{1+a_2r_0^2}+k\right]\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}$$
$$+\frac{T-\frac{t_0'-T'}{k}-t_0-2\left(1-\frac{k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}}-\frac{2}{k}\left(\frac{1+\delta^2}{1+a_2r_0^2}+k\right)\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}}{2k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)}\left[k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)\right] \quad (4.121)$$

$$t_6 = t_0 + \frac{t_0'-T'}{k} + 2\left(1-\frac{k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}} + \frac{2}{k}\left(\frac{1+\delta^2}{1+a_2r_0^2}+k\right)\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}$$
$$+\frac{T-\frac{t_0'-T'}{k}-t_0-2\left(1-\frac{k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}}-\frac{2}{k}\left(\frac{1+\delta^2}{1+a_2r_0^2}+k\right)\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}}{2k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)}\left[k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)\right] \quad (4.122)$$

$$a_1 = \frac{\delta}{r_0}\,\frac{2k+\sqrt{1+\delta^2}+\frac{1}{\delta}Ln\left(\delta+\sqrt{1+\delta^2}\right)}{kT - t_0' + T' - kt_0 - 2k\left(1-\frac{k^2}{1+a_2r_0^2}\right)\left(\frac{b_2r_0^2}{k^2-1-a_2r_0^2}\right)^{\frac{3}{2}} - 2\left(\frac{1+\delta^2}{1+a_2r_0^2}+k\right)\left(\frac{b_2r_0^2}{\delta^2-a_2r_0^2}\right)^{\frac{3}{2}}} = -a_5 \quad (4.123)$$

Deriving Equation (4.103) and making use of Equations (4.105), (4.81) at $t = t_6$, (4.107) and (4.112)

$$\frac{\partial t_V'}{\partial r} = \frac{t_1 - t_0}{r}\sqrt{1+a_1^2r^2\left(t_1-t_0\right)^2} - \frac{1}{a_1r^2}Ln\left[a_1r\left(t_1-t_0\right)+\sqrt{1+a_1^2r^2\left(t_1-t_0\right)^2}\right]$$

$$-2\left[\frac{a_2 r\left(1+a_2 r^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+a_2 b_2 r^3+3 b_2 r}{\left(1+a_2 r^2\right)^2}\right]\sqrt{\left(1+a_2 r^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2 r^2}$$

$$+2\left[\frac{a_2 r\left(1+a_2 r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+a_2 b_2 r^3+3 b_2 r}{\left(1+a_2 r^2\right)^2}\right]\sqrt{\left(1+a_2 r^2\right)\left(t_2-t_1\right)^{\frac{2}{3}}+b_2 r^2}$$

$$+\frac{a_2 r\left(t_2-t_3\right)^{\frac{2}{3}}+b_2 r}{\left(t_2-t_3\right)^{\frac{1}{3}}\sqrt{\left(1+a_2 r^2\right)\left(t_2-t_3\right)^{\frac{2}{3}}+b_2 r^2}}\left(t_4-t_3\right)+\frac{t-T}{2r}\sqrt{1+a_5{}^2 r^2\left(t-T\right)^2}$$

$$-\frac{1}{2a_5 r^2}Ln\left[a_5 r\left(t-T\right)+\sqrt{1+a_5{}^2 r^2\left(t-T\right)^2}\right] \quad (4.124)$$

The proper energy density at time $t=t_6$ is obtained from Equations (3.13), (4.82) and (4.72)

$$\rho_6=\rho(t_6)=\frac{\rho_0}{1+r^2\left(\frac{d\varphi_{IV}}{dt}\right)^2_{t=t_6}} \quad (4.125)$$

Introducing Equations (4.94), (4.96) and (4.125) into Equations (3.16) and (3.17), and making use of Equation (4.100) we obtain for this fluid in the reference frame S during the fifth leg of time travel

$$f_V^r=-\rho_0 r\frac{\left(\frac{d\varphi_V}{dt}\right)^2}{1+r^2\left(\frac{d\varphi_V}{dt}\right)^2}=-\frac{\rho_0 a_5{}^2 r\left(t-T\right)^2}{1+a_5{}^2 r^2\left(t-T\right)^2} \quad (4.126)$$

$$f_V^\theta=\rho_0\frac{r\sqrt{1+r^2\left(\frac{d\varphi_V}{dt}\right)^2}\frac{\partial t'_V}{\partial r}\left(\frac{d\varphi_V}{dt}\right)^3-\frac{d^2\varphi_V}{dt^2}}{\left[1+r^2\left(\frac{d\varphi_V}{dt}\right)^2\right]^2}=\rho_0\frac{r\sqrt{1+a_5{}^2 r^2\left(t-T\right)^2}\frac{\partial t'_V}{\partial r}a_5^3\left(t-T\right)^3-a_5}{\left[1+a_5{}^2 r^2\left(t-T\right)^2\right]^2} \quad (4.127)$$

where $\frac{\partial t'_V}{\partial r}$ is given by Equation (4.124).

The fifth leg of time travel occurs during the interval $t'_6\le t'\le T'$ for an observer located in S′. The rotating motion of this fluid in this reference frame during this leg is obtained introducing Equations (4.94) and (4.96) together with Equation (3.28) into Equations (3.29) and (3.30)

$$f_V'^r\approx-\rho_0 a_5{}^2 r\left(t'-T-t'_6+t_6\right)^2 \quad (4.128)$$

$$f_V'^\theta=-\rho_0 a_5 \quad (4.129)$$

that are valid if condition (3.25) is verified, which is right by Equations (4.112) and (4.13)

$$r\frac{d\varphi_V}{dt}=a_5 r\left(t-T\right)<a_5 r\left(t_6-T\right)=a_1 r\left(t_1-t_0\right)\ll 1 \quad (4.130)$$

And this is the end of the travel to the past.

As it is easy to imagine, the return travel to the present can be carried out in a similar way in only three legs that do not entail as much difficulty as the travel to the past.

APPENDIX: EINSTEIN'S EQUATIONS WITH SCALAR FIELD

The above results are adequate in a flat space-time in the absence of gravity. In practice, this happens when the gravitational field created by the fluid is so weak that it barely curves space-time. Although it is not relevant in the exposed developments, it is convenient to generalize the formalism to the case in which the gravitational field generated by the fluid is important. The results in the absence of gravity are obtained as a particular case. To do this, it is necessary to resort to Einstein's field equations.

First, it should be noted that the energy-momentum tensor (1.11) is not conserved, as it is deduced from Equation (1.9), so that it is not compatible with Einstein's equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu} \tag{A.1}$$

for which the energy-momentum tensor is conserved

$$T^{\mu\alpha}{}_{;\alpha} = 0 \tag{A.2}$$

This means that the energy-momentum tensor (1.11) is not adequate to calculate the gravitational field created by a perfect fluid subjected to mechanical forces.

The simplest way to avoid this difficulty is to introduce a scalar field ϕ into Einstein's Equations (A.1).
This scalar field verifies a new differential equation. The simplest generally covariant field equation for such a scalar field is

$$\phi_{;}{}^{\alpha}{}_{;\alpha} = 4\pi\lambda T^{\alpha}_{M\alpha} \tag{A.3}$$

where λ is a coupling constant and $T^{\mu\nu}_{M}$ is the energy-momentum tensor of the matter.
The most general symmetric tensor for this field must contain terms which involves two derivatives of one or two ϕ fields

$$T^{\mu\nu}_{\phi} = A(x)\phi_{;}{}^{\mu}\phi_{;}{}^{\nu} + B(x) g^{\mu\nu} \phi_{;}{}^{\alpha}\phi_{;\alpha} + C(x)\phi_{;}{}^{\mu}{}_{;}{}^{\nu} + D(x) g^{\mu\nu}\phi_{;}{}^{\alpha}{}_{;\alpha} \tag{A.4}$$

Einstein's Equations (A.1) including this scalar field are

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G\left(T_{M}{}^{\mu\nu} + T^{\mu\nu}_{\phi}\right) \tag{A.5}$$

where $T^{\mu\nu}_{M}$ is the energy-momentum tensor of the matter that for a perfect fluid is given by Equations (1.11) and $T^{\mu\nu}_{\phi}$ is the energy-momentum tensor of the scalar field given by Equations (A.4).

In order to determine the functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$ which appear in Equations (A.4), the covariant divergence of Equations (A.5) is calculated

$$\left(R^{\mu\alpha} - \frac{1}{2} g^{\mu\alpha} R\right)_{;\alpha} = -8\pi G\left(T^{\mu\alpha}_{M\ ;\alpha} + T^{\mu\alpha}_{\phi\ ;\alpha}\right) \tag{A.6}$$

Bianchi identities ensure that the first term of Equations (A.6) is null

$$\left(R^{\mu\alpha} - \frac{1}{2} g^{\mu\alpha} R\right)_{;\alpha} = 0 \tag{A.7}$$

which is equivalent with Equations (A.6) to

$$T^{\mu\alpha}_{M\ ;\alpha} = -T^{\mu\alpha}_{\phi\ ;\alpha} \tag{A.8}$$

The last equation guarantees that the energy-momentum tensor defined as

$$T^{\mu\nu} = T_M^{\ \mu\nu} + T_\phi^{\mu\nu} \tag{A.9}$$

is conserved.

As reflected in Equations (1.9) the first term of Equations (A.8) is the density of the external force

$$f^{\mu} = T^{\mu\alpha}_{M\ ;\alpha} \tag{A.10}$$

that with Equations (A.8) gives

$$f^{\mu} = -T^{\mu\alpha}_{\phi\ ;\alpha} \tag{A.11}$$

The covariant divergence of Equations (A.4) is

$$\begin{aligned} T^{\mu\gamma}_{\phi\ ;\gamma} = &\frac{\partial A}{\partial x^{\alpha}}\phi^{\mu}_{;}\phi^{\alpha}_{;} + [A(x)+2B(x)]\phi^{\mu\ \alpha}_{;\ ;}\phi_{;\alpha} + A(x)\phi^{\mu}_{;}\phi^{\alpha}_{;\ ;\alpha} + \frac{\partial B}{\partial x^{\beta}} g^{\mu\beta}\phi^{\alpha}_{;}\phi_{;\alpha} \\ &+\frac{\partial C}{\partial x^{\alpha}}\phi^{\mu\ \alpha}_{;\ ;} + C(x)\phi_{;\alpha;\ ;}^{\ \ \alpha\ \mu} + \frac{\partial D}{\partial x^{\beta}} g^{\mu\beta}\phi^{\alpha}_{;\ ;\alpha} + D(x)\phi^{\mu\ \alpha}_{;\ ;\ ;\alpha} \end{aligned} \tag{A.12}$$

where

$$\phi^{\mu\ \alpha}_{;\ ;} = \phi^{\alpha\ \mu}_{;\ ;} \tag{A.13}$$

has been used.

Introducing Equations (A.12) into Equations (A.11) gives

$$\begin{aligned} f^{\mu} = &-\frac{\partial A}{\partial x^{\alpha}}\phi^{\mu}_{;}\phi^{\alpha}_{;} - [A(x)+2B(x)]\phi^{\mu\ \alpha}_{;\ ;}\phi_{;\alpha} - A(x)\phi^{\mu}_{;}\phi^{\alpha}_{;\ ;\alpha} - \frac{\partial B}{\partial x^{\beta}} g^{\mu\beta}\phi^{\alpha}_{;}\phi_{;\alpha} \\ &-\frac{\partial C}{\partial x^{\alpha}}\phi^{\mu\ \alpha}_{;\ ;} - C(x)\phi_{;\alpha;\ ;}^{\ \ \alpha\ \mu} - \frac{\partial D}{\partial x^{\beta}} g^{\mu\beta}\phi^{\alpha}_{;\ ;\alpha} - D(x)\phi^{\mu\ \alpha}_{;\ ;\ ;\alpha} \end{aligned} \tag{A.14}$$

These are the equations of the dynamics of the scalar field that determine the four functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$.

Reference frames

The ten Equations (A.5) of the gravitational field are not independent but are related by the four Bianchi identities (A.7) which reduces Equations (A.5) to six independent equations. To determine an unambiguous metric, it is necessary to add four more equations to fix the Gauge, that is, the reference frame.

In the presence of gravity, it is also convenient to define a reference frame at rest S´ as described in section I. Since Equations (1.9) are also applicable in the case of gravity, then in the reference frame S´ the equations of motion of the fluid are

$$f'^{\mu} = T'^{\mu\alpha}_{M\ ;\alpha} = \frac{\partial T'^{\mu\alpha}_{M}}{\partial x'^{\alpha}} + \Gamma'^{\alpha}_{\alpha\beta}T'^{\mu\beta}_{M} + \Gamma'^{\mu}_{\alpha\beta}T'^{\alpha\beta}_{M} \tag{A.15}$$

where

$$T'^{\mu\nu}_{M} = pg'^{\mu\nu} + (p+\rho)\frac{dx'^{\mu}}{d\tau}\frac{dx'^{\nu}}{d\tau} \tag{A.16}$$

is the energy-momentum tensor of the fluid as in Equations (1.11) and

$$d\tau^2 = -g'_{\alpha\beta}dx'^{\alpha}dx'^{\beta} \tag{A.17}$$

is the proper time. The affine connection, as a function of the metric in S´ is obtained as in Equations (1.26)

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{1}{2} g'^{\lambda\alpha} \left(\frac{\partial g'_{\mu\alpha}}{\partial x'^{\nu}} + \frac{\partial g'_{\nu\alpha}}{\partial x'^{\mu}} - \frac{\partial g'_{\mu\nu}}{\partial x'^{\alpha}} \right) \tag{A.18}$$

In this case the metric $g'_{\mu\nu}$ satisfies Einstein's Equations (A.5)

$$R'^{\mu\nu} - \frac{1}{2} g'^{\mu\nu} R' = -8\pi G \left(T'^{\mu\nu}_{M} + T'^{\mu\nu}_{\phi} \right) \tag{A.19}$$

Clearly, in presence of gravity, the metric is different from the Minkowski metric (1.1) and the curvature tensor (1.27) is not zero, so space-time is curved and the field has sources.

Now, in the presence of gravity and as in Equations (1.2), the reference frame at rest S´ can be defined as that in which the equations of motion of the fluid are

$$f'^{\mu} = T'^{\mu\alpha}_{M\ ;\alpha} = \frac{\partial T'^{\mu\alpha}_{M}}{\partial x'^{\alpha}} \tag{A.20}$$

This equates, in view of Equations (A.15), which are valid in any reference frame and also in S´, to choose a gauge such that

$$\Gamma'^{\alpha}_{\alpha\beta} T'^{\mu\beta}_{M} + \Gamma'^{\mu}_{\alpha\beta} T'^{\alpha\beta}_{M} = 0 \tag{A.21}$$

These four equations that determine the Gauge, that is, the reference frame S´, are not covariant so they are not valid in any reference frame, but only in S´.

The reference frame in motion S is chosen, as in section I, such that the fluid, which originates the gravitational field, is at rest. Similarly, the equations of motion of the fluid in the reference frame S are

$$f^{\mu} = T^{\mu\alpha}_{M\ ;\alpha} = \frac{\partial T^{\mu\alpha}_{M}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\alpha\beta} T^{\mu\beta}_{M} + \Gamma^{\mu}_{\alpha\beta} T^{\alpha\beta}_{M} \tag{A.22}$$

where

$$T^{\mu\nu}_{M} = p g^{\mu\nu} + (p+\rho) \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \tag{A.23}$$

and

$$d\tau^2 = -g'_{\alpha\beta} dx'^{\alpha} dx'^{\beta} = -g'_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\gamma}} dx^{\delta} dx^{\gamma} = -g_{\delta\gamma} dx^{\delta} dx^{\gamma} \tag{A.24}$$

is the invariant interval. The relation between the metric in S and S´ is deduced from Equation (A.24)

$$g_{\mu\nu} = g'_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} = g_{\nu\mu} \tag{A.25}$$

Again, the affine connection is obtained as a function of the metric in S as in Equations (1.26)

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} + \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right) \tag{A.26}$$

The metric $g_{\mu\nu}$ in S also satisfies Einstein's Equations (A.5)

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G \left(T^{\mu\nu}_{M} + T^{\mu\nu}_{\phi} \right) \tag{A.27}$$

If the fluid is at rest in S, the three Equations (1.17) are valid

$$\frac{dx^i}{d\tau} = 0 \tag{A.28}$$

In addition, to fix completely the reference frame S, it is necessary to add another condition that it can be obtained as in Equation (1.18) by imposing that time-time component of the metric must be the same in any reference frame

$$g_{tt=}g'_{tt} \tag{A.29}$$

where g'_{tt} is the time-time component of the metric in the reference frame S´ at rest. The three Equations (A.28) together with Equation (A.29) fix another Gauge, that is, the reference frame S.

Degrees of freedom

In addition, the continuity equation of the fluid must be verified, that when the mass-energy in S´ is conserved takes the form

$$T'^{t\alpha}_{M\ \ ;\alpha} = 0 \tag{A.30}$$

This equation is not covariant, so it only applies in the reference frame S´.

In general, and in any reference frame, the six Einstein's independent equations with scalar field, the four equations that fix the reference frame, the four equations of motion of the fluid, the four equations of the dynamics of the scalar field, the equation of the invariant interval, the equation of state, the equation that verifies the scalar field and the continuity equation of the fluid form a system of 22 equations with 25 unknowns: the ten components of the metric tensor, the four components of the density of the external force applied, the four components of the velocity four-vector of the fluid, the four functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$, the pressure and the proper energy density of the fluid and the scalar field. This leaves 25-22=3 degrees of freedom as corresponds to a perfect fluid in motion.

In practice, it is convenient, as in Section I, that the coordinate transformations appear explicitly in the equations of the motion (A.22) in S. For this purpose the affine connection (A.26) which appears in Equations (A.22) is obtained as a function of the metric $g'_{\mu\nu}$ in S´ through Equations (A.25).

In this case the six Einstein's independent Equations (A.19), the four Equations (A.21) that fix the reference frame S´, the four equations of motion of the fluid (A.22), the four equations of the dynamics of the scalar field (A.14), the three Equations (A.28) together with the Equation (A.29), the equation of the invariant interval (A.24), the equation of state (1.16), the Equation (A.3) that verifies the scalar field and the continuity equation (A.30) form a system of 26 equations with 29 unknowns: the ten components of the metric tensor $g'_{\mu\nu}$, the four components of the density of the external force applied f^{μ}, the four components of the velocity four-vector of the fluid $\frac{dx^{\mu}}{d\tau}$, the four functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$, the four functions that relate the coordinates in both reference frames $x'^{\mu} = x'^{\mu}(\boldsymbol{x})$, the pressure p and the proper energy density of the fluid ρ and the scalar field ϕ. This again leaves 29-26=3 degrees of freedom.

These three degrees of freedom allow to choose, for example, the functions that relate the spatial coordinates in both reference frames as independent variables. The rest of variables will be expressed as functions of them.

Relevance of the scalar field

The introduction of the scalar field ϕ gives the Minkowski metric (1.1) as a solution of Einstein's Equations (A.19) in the Gauge (A.21). This solution is adequate in the absence of gravity. In practice, this happens when the gravitational field created by the fluid is so weak that it barely curves space-time. In this situation, in any reference frame, it is verified

$$R^{\mu\nu} = 0 \tag{A.31}$$

and

$$R = g_{\alpha\beta} R^{\alpha\beta} = 0 \tag{A.32}$$

With Equations (A.31), (A.32) and (A.5) it can be shown for this fluid

$$T_M^{\mu\nu} = -T_\phi^{\mu\nu} \tag{A.33}$$

so the ϕ field is especially important.

In vacuum, the energy-momentum tensor of the matter is zero

$$T_M^{\mu\nu} = 0 \tag{A.34}$$

and Equation (A.3) will be written as

$$\phi_{;\ ;\alpha}^{\ \alpha} = 0 \tag{A.35}$$

while Equations (A.10) give

$$f^{\mu} = 0 \tag{A.36}$$

Introducing Equations (A.35) and (A.36) into Equations (A.14) we obtain

$$0 = \frac{\partial A}{\partial x^\alpha} \phi_;^{\ \mu} \phi_;^{\ \alpha} + \left[A(x) + 2B(x) \right] \phi_{;\ ;}^{\ \mu\ \alpha} \phi_{;\alpha} + \frac{\partial B}{\partial x^\beta} g^{\mu\beta} \phi_;^{\ \alpha} \phi_{;\alpha} + \frac{\partial C}{\partial x^\alpha} \phi_{;\ ;}^{\ \mu\ \alpha} + C(x) \phi_{;\alpha;\ ;}^{\ \ \ \alpha\ \mu} \tag{A.37}$$

The solutions of these equations are

$$A(x) = B(x) = C(x) = 0 \tag{A.38}$$

With Equations (A.35) and (A.38) the energy-momentum tensor (A.4) of the scalar field ϕ in vacuum is

$$T_\phi^{\mu\nu} = 0 \tag{A.39}$$

Equations (A.34) and (A.39) imply that Equations (A.1) and (A.5) have the same solutions in vacuum, that is, in this case the scalar field does not alter the solutions of the Equations (A.1) that explain, among other phenomena, the dynamics of the planets. The same happens when the density of the external force applied is zero

$$f^{\mu} = 0 \tag{A.40}$$

since in this situation Equations (A.14) provide

$$A(x) = B(x) = C(x) = D(x) = 0 \tag{A.41}$$

which with Equations (A.4) lead again to Equations (A.39) and the ϕ field having no influence. In other cases, their significance will have to be determined.

Cosmological dark energy

Finally, it should be noted that in the absence of gravity or weak fields, the only way to produce acceleration over the fluid is, as it has been shown, by the application of external forces f^{μ}. However, when the gravity generated by the fluid is strong, it can alone accelerate the fluid. This acceleration causes a force f^{μ} as described above, which produces the same effects, but it has a purely gravitational origin. This is because the force f^{μ} appears whenever inertial forces act on the fluid. In other words, when the gravitational field generated by the fluid is strong, the energy-momentum tensor (1.11) of the matter is not conserved. In this way, the scalar field ϕ can be present even if only gravitational forces act on the fluid.

This has important cosmological implications: the scalar field ϕ appears in Cosmology and the cosmological dark energy has its origin in the term $T_{\phi}^{\mu\nu}$ of Equations (A.5). In this case, three of the four functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$, which appear in Equations (A.4), are independent and they can be adjusted to the observational data of accelerated expansion of the Universe.

CONCLUSIONS

The exposed developments demonstrate the theoretical possibility of travelling to the past, in the case of a macroscopic perfect fluid without pressure in the absence of gravity and without violating the laws of Special Relativity. This is only possible if the fluid is accelerated to the speed of light. To demonstrate this, a relativistic treatment of rotation has been made using the principle of General Covariance, which has proved to be practical. Finally, the need to introduce a scalar field in Einstein's equations has been justified to explain relativistic dynamics satisfactorily. These equations have made it possible to generalize the formalism to the case in which the fluid generates an appreciable Gravitational Field.

And to finish, I will say just a few words: "only time will tell the truth".

BIBLIOGRAPHY

S. Weinberg: "Gravitation and Cosmology", Wiley, New York, 1972, Ch. 2 to 7.

REFERENCES

[1] A.P. French, Special Relativity, Massachusetts Institute of Technology, 1968, p. 71